

Möbius characterization of the boundary at infinity of rank one symmetric spaces

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Abstract

A Möbius structure (on a set X) is a class of metrics having the same cross-ratios. A Möbius structure is ptolemaic if it is invariant under inversion operations. The boundary at infinity of a $\text{CAT}(-1)$ space is in a natural way a Möbius space, which is ptolemaic. We give a free of classification proof of the following result that characterizes the rank one symmetric spaces of noncompact type purely in terms of their Möbius geometry: Let X be a compact Ptolemy space which contains a Ptolemy circle and allows many space inversions. Then X is Möbius equivalent to the boundary at infinity of a rank one symmetric space.

1 Introduction

Two metrics on a set X are Möbius equivalent if they have the same cross-ratios. A Möbius structure on X is a class of Möbius equivalent metrics. If a Möbius structure is fixed then X is called a Möbius space. Ptolemy spaces are Möbius spaces with the property that the inversion operation preserves the Möbius structure. A classical example of a Ptolemy space is the extended $\widehat{\mathbb{R}}^n = \mathbb{R}^n \cup \infty = S^n$, $n \geq 0$, where the Möbius structure is generated by an Euclidean metric on \mathbb{R}^n , and $\widehat{\mathbb{R}}^n$ is identified with the unit sphere $S^n \subset \mathbb{R}^{n+1}$ via the stereographic projection. For more detail see Section 2.

There is a well known deep connection between the geometry of the hyperbolic space H^{n+1} and the Möbius geometry of its boundary $\partial_\infty H^{n+1} = \widehat{\mathbb{R}}^n$. More generally the boundary $X = \partial_\infty Y$ of a $\text{CAT}(-1)$ space Y carries in a natural way a Möbius structure and is actually a Ptolemy space [FS1]. An isometry of Y induces a Möbius map of X and a Ptolemy circle σ in X corresponds to a totally geodesic subspace $Y_\sigma \subset Y$ isometric to H^2 with $\partial_\infty Y_\sigma = \sigma$. Here a Ptolemy circle is a subspace Möbius equivalent to the Ptolemy space $\widehat{\mathbb{R}} = S^1$.

*Supported by RFBR Grant 11-01-00302-a and SNF Grant 20-119907/1

†Supported by Swiss National Science Foundation Grant 20-119907/1

Our motivation is to find a Möbius characterization of the boundary at infinity of rank one symmetric spaces Y of non-compact type. In the case $Y = \mathbb{H}^{n+1}$ this problem is solved in [FS2] for every $n \geq 1$: *every compact Ptolemy space such that through any three points there is a Ptolemy circle is Möbius equivalent to $\widehat{\mathbb{R}}^n = \partial_\infty \mathbb{H}^{n+1}$.*

Given distinct points ω, ω' in a Möbius space X , there is a well defined notion of a *sphere* S between ω, ω' , see sect. 2.3, and a notion of a *space inversion* w.r.t. ω, ω', S , which is a fixed point free Möbius involution $\varphi_{\omega, \omega', S} : X \rightarrow X$, permuting ω and ω' , preserving S and any Ptolemy circle through ω, ω' , see sect. 3.1.

We consider a Möbius space X with the following basic properties.

- (E) Existence: there is at least one Ptolemy circle in X .
- (I) Inversion: for each distinct $\omega, \omega' \in X$ and every sphere $S \subset X$ between ω, ω' there is a unique space inversion $\varphi_{\omega, \omega', S} : X \rightarrow X$ w.r.t. ω, ω' and S .

Our main result gives the following Möbius characterization of the boundary at infinity of rank one symmetric spaces of non-compact type.

Theorem 1.1. *Let X be a compact Ptolemy space with properties (E) and (I). Then X is Möbius equivalent to the boundary at infinity of a rank one symmetric space of non-compact type taken with the canonical Möbius structure.*

The canonical Möbius structure on the boundary at infinity $X = \partial_\infty Y$ of every rank one symmetric space Y of non-compact type can be described as follows. Assume that the metric of Y is normalized such that maximum of sectional curvatures equals -1 . Take $\omega \in X$ and a Busemann function $b : Y \rightarrow \mathbb{R}$ centered at ω . Then the function $d_b : X \times X \rightarrow \mathbb{R}$ defined by

$$d_b(\xi, \xi') = e^{-(\xi|\xi')_b},$$

where $(\xi|\xi')_b \in \mathbb{R}$ is the Gromov product of ξ, ξ' w.r.t. b , is a metric on X . The canonical Möbius structure on X is generated by all such metrics d_b with $b \in \omega \in X$.

Remark 1.2. The property (E) plays a double role. First, it put a restriction on the topology of X excluding e.g. totally disconnected spaces. Second, it also serves as a normalization condition, because, for example, the Ptolemy space $X = \partial_\infty \mathbb{H}^{n+1}$, where \mathbb{H}^{n+1} is a real hyperbolic space of curvature -1 , contains a lot of Ptolemy circles, while for every $0 < \lambda < 1$ the boundary at infinity X_λ of the rescaled hyperbolic space $\lambda \mathbb{H}^{n+1}$ is still a Ptolemy space which however contains no Ptolemy circle. This is because the rescaling $\mathbb{H}^{n+1} \mapsto \lambda \mathbb{H}^{n+1}$ implies the operation $d \mapsto d^\lambda$ for every metric d of the Möbius structure of X .

The proof of Theorem 1.1 relies on two existence results. The first one, which is formulated as the property (K) in Theorem 3.4, provides for every

$\omega \in X$ the existence of a canonical fibration $\pi_\omega : X_\omega \rightarrow B_\omega$ and for every $x \in X_\omega$ the existence of a Ptolemy circle through x, ω , which hits a prescribed fiber of π_ω ; here $X_\omega = X \setminus \omega$. Preparation to the proof of this result, the proof itself and discussion of various consequences occupy the first part of the paper, sections 3 – 6. Most important consequence obtained here is the existence of a simply connected, nilpotent Lie group structure N_ω on every punctured space X_ω , $\omega \in X$, see sect. 6. At this point, one can formally conclude the proof of Theorem 1.1 referring to the Tits classification of 2-transitive group actions, see [Kr]. The Tits classification gives an alternative between two possibilities. One of them is related to the rank one symmetric space on non-compact type, and the first part of the paper is actually the proof that the other one cannot be realized as Möbius transformations of any Möbius structure on X .

However, we are not satisfied with that formal classification argument, and we prefer to give a direct, classification free proof of Theorem 1.1 to clarify the geometry of the phenomenon. We directly construct a symmetric space which has the Möbius space X as boundary at infinity. This is done in the second part of the paper, sections 7 – 8: We introduce a filling $Y = \text{Fill } X$ of X as the set of all space inversions of X and a distance ρ on Y defined via cross-ratios of the Möbius structure of X . Then we show that ρ is a Riemannian distance associated with a Riemannian rank one symmetric space of non-compact type with maximum of sectional curvatures -1 . The proof is based on our second existence result that every Möbius map between two Ptolemy circles in X extends to a Möbius automorphism of X , see sect. 7. Finally, we conclude the proof of Theorem 1.1 by showing that $X = \partial_\infty Y$ and that the canonical Möbius structure associated with Y coincides with the initial one on X .

Our proof of Theorem 1.1 neither uses the classification of rank one symmetric spaces of non-compact type, nor gives a new approach to that classification.

Section 2 serves as a brief introduction to Möbius geometry.

Acknowledgments. We are thankful to L. Kramer for informing us about 2-transitive group actions. The first author is also grateful to the University of Zürich for hospitality and support.

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2 Möbius structures and Ptolemy spaces

This section is a brief introduction to Möbius geometry.

2.1 Möbius structures

A quadruple $Q = (x, y, z, u)$ of points in a set X is said to be *admissible* if no entry occurs three or four times in Q . Two metrics d, d' on X are *Möbius equivalent* if for any admissible quadruple $Q = (x, y, z, u) \subset X$ the respective *cross-ratio triples* coincide, $\text{crt}_d(Q) = \text{crt}_{d'}(Q)$, where

$$\text{crt}_d(Q) = (d(x, y)d(z, u) : d(x, z)d(y, u) : d(x, u)d(y, z)) \in \mathbb{RP}^2.$$

We actually consider *extended* metrics on X for which existence of an *infinitely remote* point $\omega \in X$ is allowed, that is, $d(x, \omega) = \infty$ for all $x \in X$, $x \neq \omega$. We always assume that such a point is unique if exists, and that $d(\omega, \omega) = 0$. We use notation $X_\omega := X \setminus \omega$ and the standard conventions for the calculation with $\omega = \infty$. If ∞ occurs once in Q , say $u = \infty$, then $\text{crt}_d(x, y, z, \infty) = (d(x, y) : d(x, z) : d(y, z))$. If ∞ occurs twice, say $z = u = \infty$, then $\text{crt}_d(x, y, \infty, \infty) = (0 : 1 : 1)$.

A *Möbius structure* on a set X is a class $\mathcal{M} = \mathcal{M}(X)$ of metrics on X which are pairwise Möbius equivalent.

The topology considered on (X, d) is the topology with the basis consisting of all open distance balls $B_r(x)$ around points in $x \in X_\omega$ and the complements $X \setminus D$ of all closed balls $D = \overline{B}_r(x)$. Möbius equivalent metrics define the same topology on X . When a Möbius structure \mathcal{M} on X is fixed, we say that (X, \mathcal{M}) or simply X is a *Möbius space*.

A map $f : X \rightarrow X'$ between two Möbius spaces is called *Möbius*, if f is injective and for all admissible quadruples $Q \subset X$

$$\text{crt}(f(Q)) = \text{crt}(Q),$$

where the cross-ratio triples are taken with respect to some (and hence any) metric of the Möbius structure of X and of X' . Möbius maps are continuous. If a Möbius map $f : X \rightarrow X'$ is bijective, then f^{-1} is Möbius, f is homeomorphism, and the Möbius spaces X, X' are said to be *Möbius equivalent*.

In general different metrics in a Möbius structure \mathcal{M} can look very differently. However if two metrics have the same infinitely remote point, then they are homothetic. Since this result is crucial for our considerations, we state it as a lemma.

Lemma 2.1. *Let \mathcal{M} be a Möbius structure on a set X , and let $d, d' \in \mathcal{M}$ have the same infinitely remote point $\omega \in X$. Then there exists $\lambda > 0$ such that $d'(x, y) = \lambda d(x, y)$ for all $x, y \in X$.*

Proof. Since otherwise the result is trivial, we can assume that there are distinct points $x, y \in X_\omega$. Take $\lambda > 0$ such that $d'(x, y) = \lambda d(x, y)$. If $z \in X_\omega$, then $\text{crt}_d(x, y, z, \omega) = \text{crt}_{d'}(x, y, z, \omega)$, hence $(d'(x, y) : d'(x, z) : d'(y, z)) = (d(x, y) : d(x, z) : d(y, z))$. Since $d'(x, y) = \lambda d(x, y)$ we therefore obtain $d'(x, z) = \lambda d(x, z)$ and $d'(y, z) = \lambda d(y, z)$. \square

In what follows we always consider $X_\omega = X \setminus \omega$ as a metric space with a metric from the Möbius structure for which the point ω is infinitely remote.

A classical example of a Möbius space is the extended $\widehat{\mathbb{R}}^n = \mathbb{R}^n \cup \infty = S^n$, $n \geq 1$, where the Möbius structure is generated by some extended Euclidean metric on $\widehat{\mathbb{R}}^n$, and $\mathbb{R}^n \cup \infty$ is identified with the unit sphere $S^n \subset \mathbb{R}^{n+1}$ via the stereographic projection. Note that Euclidean metrics which are not homothetic to each other generate different Möbius structures by the lemma above, which however are Möbius equivalent.

2.2 Ptolemy spaces

A Möbius space X is called a *Ptolemy space*, if it satisfies the Ptolemy property, that is, for all admissible quadruples $Q \subset X$ the entries of the respective cross-ratio triple $\text{crt}(Q) \in \mathbb{R}P^2$ satisfies the triangle inequality.

The importance of the Ptolemy property comes from the following fact. Given a metric $d \in \mathcal{M}(X)$ possibly with infinitely remote point $\omega \in X$ and a point $z \in X_\omega$, the *metric inversion*, or *m-inversion* for brevity, of d of radius $r > 0$ with respect to z is a function $d_z(x, y) = \frac{r^2 d(x, y)}{d(z, x)d(z, y)}$ for all $x, y \in X$ distinct from z , $d_z(x, z) = \infty$ for all $x \in X \setminus \{z\}$ and $d_z(z, z) = 0$. In particular, z is infinitely remote for d_z . Using the standard convention we also have $d_z(x, \omega) = \frac{r^2}{d(x, z)}$. A direct computation shows that d_z is Möbius equivalent to d .

Remark 2.2. When saying about an m-inversion of a metric without specifying its radius, we mean that the radius is 1.

In general d_z is not a metric because the triangle inequality may not be satisfied. However, we have

Proposition 2.3. *A Möbius structure \mathcal{M} on a set X is Ptolemy if and only if \mathcal{M} is invariant under the metric inversion $d \mapsto d_z$ w.r.t. every $z \in X$.*

Proof. Since d_z is Möbius equivalent to d

$$(d_z(x, y) : d_z(y, u) : d_z(x, u)) = \text{crt}_{d_z}(x, y, z, u) = \text{crt}_d(x, y, z, u)$$

for $x, y, u \in X \setminus z$. Thus the triangle inequality for d_z is equivalent to the Ptolemy property of d . \square

The classical example of Ptolemy space is $\widehat{\mathbb{R}}^n$ with a standard Möbius structure as it follows from the proposition above. Here is the list of some known results on metric spaces with Ptolemy property. A real normed vector space, which is ptolemaic, is an inner product space (Schoenberg, 1952, [Sch]); a Riemannian locally ptolemaic space is nonpositively curved (Kay, 1963, [Kay]); all Bourdon and Hamenstädt metrics on $\partial_\infty Y$, where Y is CAT(−1), generate a Ptolemy space (Foertsch-Schroeder, 2006, [FS1]); a

geodesic metric space is CAT(0) if and only if it is ptolemaic and Busemann convex, a ptolemaic proper geodesic metric space is uniquely geodesic (Foertsch-Lytchak-Schroeder, 2007, [FLS]); any Hadamard space ptolemaic, a complete Riemannian manifold is ptolemaic if and only if it is a Hadamard manifold, a Finsler ptolemaic manifold is Riemannian (Buckley-Falk-Wraith, 2009, [BFW]). These results allow to suggest that the Ptolemy property is a sort of a Möbius invariant nonpositive curvature condition.

2.3 Spheres between points

The notion of a sphere between two points is a notion of Möbius geometry and can be described in terms of the cross-ratio triple. Let X be a Möbius space. Given distinct $\omega, \omega' \in X$, we say that $x, y \in X$ lie on some sphere between ω, ω' if $\text{crt}(\omega, x, y, \omega') = (1 : 1 : *)$, i.e. the first two entries are equal. In particular, both x, y are distinct from ω, ω' . One easily checks this defines an equivalence relation on $X \setminus \{\omega, \omega'\}$, and any equivalence class $S \subset X \setminus \{\omega, \omega'\}$ is called a *sphere between* ω, ω' . Möbius maps preserve the sets of spheres between points: if $\varphi : X \rightarrow X$ is a Möbius map, and S is a sphere between ω and ω' , then $\varphi(S)$ is a sphere between $\varphi(\omega)$ and $\varphi(\omega')$.

If ω is infinitely remote for some metric d of the Möbius structure then

$$S = \{x \in X : d(x, \omega') = r\} = S_r^d(\omega')$$

for some $r > 0$, which justifies our terminology.

Given distinct points $x, y, \omega, \omega' \in X$ the symmetries of the cross-ratio triple implies that x and y lie on a some sphere between ω and ω' if and only if ω and ω' lie on a some sphere between x and y .

If we take some point on the sphere as infinitely remote, then the sphere becomes a bisector w.r.t. a respective metric.

Lemma 2.4. *Let $S \subset X$ be a sphere between distinct $a, a' \in X$. Then for every $\omega \in S$ the set $S_\omega = S \setminus \{\omega\}$ is the bisector in X_ω between a, a' , $S_\omega = \{x \in X_\omega : d(x, a) = d(x, a')\}$.*

Proof. In the space X_ω we have $\text{crt}(a, \omega, x, a') = (d(x, a') : d(x, a) : *)$. Hence, $x \in S_\omega$ if and only if $d(x, a) = d(x, a')$. \square

2.4 Circles in Ptolemy spaces

A Ptolemy circle in a Möbius space X is a subset $\sigma \subset X$ homeomorphic to S^1 such that for every quadruple $(x, y, z, u) \in \sigma$ of distinct points the equality

$$d(x, z)d(y, u) = d(x, y)d(z, u) + d(x, u)d(y, z) \quad (1)$$

holds for some and hence for any metric d of the Möbius structure, where it is supposed that the pair (x, z) separates the pair (y, u) , i.e. y and u are in

different components of $\sigma \setminus \{x, z\}$. Recall the classical Ptolemy theorem that four points x, y, z, u of the Euclidean plane lie on a circle (in this order) if and only if their distances satisfy the Ptolemy equality (1).

Let σ be a Ptolemy circle passing through the infinitely remote point ω for some metric $d \in \mathcal{M}$ and let $\sigma_\omega = \sigma \setminus \omega$. Then (1) says that for $x, y, z \in \sigma_\omega$ (in this order) $d(x, y) + d(y, z) = d(x, z)$, i.e. it implies that σ_ω is a geodesic, actually a complete geodesic isometric to \mathbb{R} .

We recall the following fact from [FS2].

Proposition 2.5. *Let σ and σ' be Ptolemy circles. Let x_1, x_2, x_3 and x'_1, x'_2, x'_3 be distinct points on σ respectively on σ' . Then there exists a unique Möbius homeomorphism $\varphi : \sigma \rightarrow \sigma'$ with $\varphi(x_i) = x'_i$. \square*

In particular all Ptolemy circles are Möbius equivalent. The standard metric models of a circle are $(\widehat{\mathbb{R}}, d)$, where d is the standard Euclidean metric, or (S^1, d_c) , where d_c is the chordal metric on S^1 , i.e. the metric induced by the standard embedding $S^1 \subset \mathbb{R}^2$ as a unit circle. These two standard realizations of a circle are Möbius equivalent via the stereographic projection. Note that by Lemma 2.1 there is up to homothety only one metric on a circle with a infinitely remote point, while there are plenty of bounded metrics (for a description of all Ptolemy metrics on S^1 see [FS2]).

2.5 Duality between Busemann and distance functions

Let X be a Ptolemy space, d a metric of the Möbius structure with infinitely remote point ω , $X_\omega = X \setminus \omega$. If a Ptolemy circle $\sigma \subset X$ passes through ω , then $l = \sigma_\omega$ is isometric w.r.t. d to a geodesic line and it is called a *Ptolemy line* in X_ω .

With every oriented Ptolemy line $l \subset X_\omega$ and every point $\omega' \in l$ we associate a function $b : X_\omega \rightarrow \mathbb{R}$, called a *Busemann function* of l , as follows. Given $x \in X_\omega$, the difference $d(x, y) - d(\omega', y)$ is nonincreasing by triangle inequality as $y \in l$ goes to infinity according the orientation of l , $y > \omega'$. Thus the limit $b(x) = \lim_{l \ni y \rightarrow \infty} (d(x, y) - d(\omega', y))$ exists. Note that $b(\omega') = 0$ and $b(x) = -d(\omega', x)$ for all $l \ni x > \omega'$.

For any Ptolemy space X there is a remarkable duality between Busemann and distance functions which is described as follows.

On a Ptolemy line $l \subset X_\omega$, we fix $\omega' \in l$, and let d' be the m-inversion of d w.r.t. ω' . Then d' is a metric of the Möbius structure with infinitely remote point ω' . In particular, $l' = \sigma_{\omega'}$ is a Ptolemy line in $X_{\omega'}$. One easily checks that d is the m-inversion of d' w.r.t. ω , that is, the inversion operation is involutive.

Let $c : \mathbb{R} \rightarrow X_{\omega'}$ be a unit speed parameterization of l' with $c(0) = \omega$, $b^\pm : X_\omega \rightarrow \mathbb{R}$ the *opposite* Busemann functions of l , that is, associated with opposite ends of l , which are normalized by $b^\pm(\omega') = 0$ and $b^+ \circ c(t) < 0$

for $t > 0$, $b^- \circ c(t) < 0$ for $t < 0$. Since $d(x, \omega') \cdot d'(x, \omega) = 1$ for every $x \in X \setminus \{\omega, \omega'\}$, we have $b^\pm \circ c(t) = \mp 1/t$ for all $t \neq 0$.

Proposition 2.6. *For all $x \in X \setminus \{\omega, \omega'\}$ we have*

$$b^\pm(x) = \frac{d^\pm}{dt} \ln d'(x, c(t))|_{t=0}, \quad (2)$$

where $\frac{d^\pm}{dt}$ is the right/left derivative.

Remark 2.7. Note that the left hand side of (2) is computed in the space X_ω while the right hand side in the inverted space $X_{\omega'}$. The equality (2) is our first example of duality equalities which appear in different places of the paper, see e.g. Remark 4.3.

Proof. We first note that the function $t \mapsto d'(x, c(t))$ is convex by the Ptolemy condition, and thus it has the right and the left derivatives at every point. Hence, the right hand side of Equation (2) is well defined. By definition, $d(x, y) = \frac{d'(x, y)}{d'(\omega, x)d'(\omega, y)}$ and $d(x, \omega') = \frac{1}{d'(x, \omega)}$ for all $x, y \in X_\omega$. Now, we compute

$$\begin{aligned} d(x, c(t)) - d(\omega', c(t)) &= \frac{d'(x, c(t))}{d'(\omega, x)d'(\omega, c(t))} - \frac{1}{d'(\omega, c(t))} \\ &= \frac{1}{|t|d'(x, c(0))} (d'(x, c(t)) - d'(x, c(0))) \end{aligned}$$

for all $t \neq 0$, because $d'(x, \omega) = d'(x, c(0))$ and $d'(\omega, c(t)) = |t|$. Since $b^\pm(x) = \lim_{t \rightarrow \pm 0} (d(x, c(t)) - d(\omega', c(t)))$, we obtain

$$b^\pm(x) = \frac{d^\pm}{dt} \ln d'(x, c(t))|_{t=0}.$$

□

Given a Ptolemy circle $\sigma \in X$ and distinct points $\omega, \omega' \in \sigma$, we denote by $D_{\sigma, \omega}^{\omega'}$ the subset in $X_{\omega'}$ which consists of all x such that ω is a closest to x point in the geodesic line $\sigma_{\omega'}$ (w.r.t. the metric of $X_{\omega'}$).

Lemma 2.8. *Let X be a Ptolemy space. Then for every Ptolemy circle $\sigma \subset X$ and each pair of distinct points $\omega, \omega' \in \sigma$ we have*

$$D_{\sigma, \omega}^{\omega'} \cup \omega' = B_{\sigma, \omega'}^\omega \cup \omega, \quad (3)$$

where $B_{\sigma, \omega'}^\omega = \{x \in X_\omega : b^+(x) \geq 0 \text{ and } b^-(x) \geq 0\}$, $b^\pm : X_\omega \rightarrow \mathbb{R}$ are the opposite Busemann functions of the Ptolemy line $\sigma_\omega \subset X_\omega$ with $b^\pm(\omega') = 0$.

Proof. Denote by d' the metric of $X_{\omega'}$ and let $c : \mathbb{R} \rightarrow X_{\omega'}$ be the unit speed parameterization of the Ptolemy line $\sigma_{\omega'} \subset X_{\omega'}$ such that $c(0) = \omega$ and $b^\pm \circ c(t) = \mp 1/t$, see the paragraph preceding Proposition 2.6. For every $x \in D_{\sigma, \omega}^{\omega'}$ we have $\frac{d^+}{dt} d'(x, c(t))_{t=0} \geq 0$ for the right derivative, and $-\frac{d^-}{dt} d'(x, c(t))_{t=0} \leq 0$ for left derivative because $t = 0$ is a minimum point of the convex function $t \mapsto d'(x, c(t))$. Equation (2) implies that $x \in B_{\sigma, \omega'}^\omega$.

Assume that $b^+(x) \geq 0$ and $b^-(x) \geq 0$ for some $x \in X \setminus \{\omega, \omega'\}$. Equation (2) implies that the right derivative $\frac{d^+}{dt} d'(x, c(t))_{t=0} \geq 0$ and the left derivative $-\frac{d^-}{dt} d'(x, c(t))_{t=0} \leq 0$. Thus $t = 0$ is a minimum point of the convex function $t \mapsto d'(x, c(t))$ and hence $x \in D_{\sigma, \omega}^{\omega'}$. \square

2.6 Busemann flat Ptolemy spaces

A Ptolemy space X is said to be (*Busemann*) *flat* if for every Ptolemy circle $\sigma \subset X$ and every point $\omega \in \sigma$, we have

$$b^+ + b^- \equiv \text{const} \quad (4)$$

for opposite Busemann functions $b^\pm : X_\omega \rightarrow \mathbb{R}$ associated with Ptolemy line σ_ω .

Busemann flatness implies that Busemann functions are affine in the following sense.

Proposition 2.9. *Let X be a Busemann flat ptolemaic space and let $l, l' \in X_\omega$ be Ptolemy lines. Then the Busemann functions of l are affine functions on l' .*

Proof. The opposite Busemann functions b^+, b^- of l are convex on l' , see [FS2]. Since the sum $b^+ + b^-$ is affine on l' by Busemann flatness, b^+ and b^- are affine. \square

The property of Busemann flatness is equivalent to that any horospheres of b^+, b^- coincide whenever they have a common point. Thus the horosphere $H_{\sigma, \omega'}^\omega \subset X_\omega$ of σ_ω through $\omega' \in \sigma_\omega$ is well defined in a flat Ptolemy space.

Proposition 2.10. *A Ptolemy space X is flat if and only if for every $\omega \in X$ and every $x \in X_\omega$ the distance function $d(x, \cdot)$ is C^1 -smooth along any Ptolemy line $l \subset X_\omega, l \not\ni x$.*

Proof. Assume that distance functions are C^1 -smooth along Ptolemy lines. We fix $\omega \in X$, a Ptolemy line $l \subset X_\omega$, and let b^\pm be opposite Busemann functions of l . We suppose W.L.G. that $b^\pm(\omega') = 0$ for some point $\omega' \in l$. Then $b^+ + b^- = 0$ along l . Equation (2) implies that in fact $b^+(x) + b^-(x) = 0$ for every $x \in X_\omega$. Thus X is flat.

Conversely, assume that X is flat. Given $\omega' \in X$, a Ptolemy line $l' \in X_{\omega'}$ and $x \in X_{\omega'} \setminus l'$, we show that the distance function $d'(x, \cdot)$ in $X_{\omega'}$ is C^1 -smooth along l' at every point $\omega \in l'$.

Let $c : \mathbb{R} \rightarrow X_{\omega'}$ by a unit speed parameterization of l' with $c(0) = \omega$, $b^\pm : X_\omega \rightarrow \mathbb{R}$ the opposite Busemann function associated with the Ptolemy line $l = (l' \cup \omega') \setminus \omega \subset X_\omega$ such that $b^\pm(\omega') = 0$, $b^+ \circ c(t) < 0$ for all $t > 0$. Then $b^+ + b^- \equiv 0$ by the assumption, and by Proposition 2.6 we have $\frac{d^+}{dt}d'(x, c(t))|_{t=0} = -\frac{d^-}{dt}d'(x, c(t))|_{t=0}$, where $\frac{d^+}{dt}$ is the right derivative and $-\frac{d^-}{dt}$ is the left derivative. Hence $d'(x, \cdot)$ is C^1 -smooth. \square

By Proposition 2.10, the duality equation (2) in a flat Ptolemy space X takes the following form

$$b^\pm(x) = \pm \frac{d}{dt} \ln d'(x, c(t))|_{t=0}. \quad (5)$$

Example 2.11. The Ptolemy space $\widehat{H}^n = H^n \cup \{\infty\}$, $n \geq 2$, generated by the real hyperbolic space H^n , is not flat because the equality $b^+ + b^- \equiv \text{const}$ is violated in H^n . (Recall that H^n possesses the Ptolemy property and thus it generates a Ptolemy space by taking all metrics on \widehat{H}^n which are Möbius equivalent to the metric of H^n .) Note that the distance function $d(x, \cdot) : H^n \rightarrow \mathbb{R}$ is smooth for every $x \in H^n$ along any geodesic line l , $x \notin l \subset H^n$. This does not contradict Proposition 2.10 because the m-inversion of d with respect to any point $x \in H^n$ has a singularity at the infinity point of \widehat{H}^n .

In flat Ptolemy spaces, the duality between distance and Busemann functions takes the following form.

Lemma 2.12. *Let X be a flat Ptolemy space, $\sigma \subset X$ a Ptolemy circle, and $\omega, \omega' \subset \sigma$ distinct points. Let $H_{\sigma, \omega'}^\omega \subset X_\omega$ be the horosphere through ω' of the Ptolemy line $\sigma_\omega \subset X_\omega$, $D_{\sigma, \omega}^{\omega'} \subset X_{\omega'}$ the set of all $x \in X_{\omega'}$ such that ω is the closest to x point in the Ptolemy line $\sigma_{\omega'}$. Then*

$$H_{\sigma, \omega'}^\omega \cup \omega = D_{\sigma, \omega}^{\omega'} \cup \omega'. \quad (6)$$

Proof. In a flat Ptolemy space we have $H_{\sigma, \omega'}^\omega = B_{\sigma, \omega'}^\omega$ because level sets of opposite Busemann functions associated with a Ptolemy line coincide when they have a common point. On the other hand, by duality, Lemma 2.8, we have $B_{\sigma, \omega'}^\omega \cup \omega = D_{\sigma, \omega}^{\omega'} \cup \omega'$. \square

3 Möbius spaces with circles and many space inversions

We begin this section with discussion of what is a space inversion of an arbitrary Möbius space.

3.1 Space inversions

We define a *space inversion*, or s-inversion for brevity, w.r.t. distinct $\omega, \omega' \in X$ and a sphere $S \subset X$ between ω, ω' as a Möbius involution $\varphi = \varphi_{\omega, \omega', S} : X \rightarrow X$, $\varphi^2 = \text{id}$, without fixed points such that

- (1) $\varphi(\omega) = \omega'$ (and thus $\varphi(\omega') = \omega$);
- (2) φ preserves S , $\varphi(S) = S$;
- (3) $\varphi(\sigma) = \sigma$ for any Ptolemy circle $\sigma \subset X$ through ω, ω' .

Remark 3.1. Motivation of this definition comes from the fact that in the case $X = \partial_\infty Y$, where Y is a symmetric rank one space of non-compact type, any central symmetry $f : Y \rightarrow Y$ with a center $o \in Y$, $f(o) = o$, induces a space inversion $\partial_\infty f = \varphi_{\omega, \omega', S} : X \rightarrow X$, where a geodesic line $l = (\omega, \omega') \subset Y$ with the end points ω, ω' passes through o , and $S \subset X$ is a sphere between ω, ω' .

Remark 3.2. In general, there is no reason that an s-inversion $\varphi = \varphi_{\omega, \omega', S}$ is uniquely determined by its data ω, ω', S . However, if φ' is another s-inversion with the same data, then it coincides with φ along any Ptolemy circle through ω, ω' because any Möbius automorphism of a Ptolemy circle is uniquely determined by values at three distinct points, see Proposition 2.5.

A Möbius automorphism $\varphi : X \rightarrow X$ of a Möbius space induces a map $\varphi^* : \mathcal{M} \rightarrow \mathcal{M}$, $(\varphi^*d)(x, y) = d(\varphi(x), \varphi(y))$ for every metric $d \in \mathcal{M}$ and each $x, y \in X$, where \mathcal{M} is the Möbius structure of X . Note that a metric inversion of a bounded metric cannot be induced by any Möbius automorphism $X \rightarrow X$, because a metric inversion w.r.t. $\omega \in X$ has ω as the infinitely remote point.

However if φ is a Möbius automorphism with $\varphi(\omega) = \omega'$ and d is a metric with infinite remote point ω , then ω' is infinitely remote for φ^*d . Thus by Lemma 2.1 $\varphi^*d = \lambda d'$ for some $\lambda > 0$, where d' is the m-inversion of d w.r.t. ω' , i.e.

$$(\varphi^*d)(x, y) = \frac{\lambda d(x, y)}{d(x, \omega')d(y, \omega')}$$

for each $x, y \in X$ which are not equal to ω' simultaneously.

In the case that $\varphi = \varphi_{\omega, \omega', S}$ an s-inversion we can say in addition that

Lemma 3.3. *We have*

$$(\varphi^*d)(x, y) = \frac{r^2 d(x, y)}{d(x, \omega')d(y, \omega')}$$

where $r = r(d)$ is determined by the equation $S = S_r^d(\omega')$.

Proof. We already know that $(\varphi^*d)(x, y) = \frac{\lambda d(x, y)}{d(x, \omega')d(y, \omega')}$ for each $x, y \in X$ which are not equal to ω , simultaneously. We compute λ by taking $x \in S$, $y = \varphi(x)$. Then $\varphi(y) = x$ and since $(\varphi^*d)(x, y) = d(x, y)$, $d(x, \omega') = r = d(y, \omega')$, we have $\lambda = r^2$. \square

Contrary to metric inversions which always exist, in general there is no reason for a space inversion to exist. The existence of many space inversions put severe restrictions on the space. In the following theorem we recover some geometric properties of a space satisfying the assumptions of Theorem 1.1. Recall the basic properties, see Introduction

(E) Existence: there is at least one Ptolemy circle in X .

(I) Inversion: for each distinct $\omega, \omega' \in X$ and every sphere $S \subset X$ between ω, ω' there is a unique space inversion $\varphi_{\omega, \omega', S} : X \rightarrow X$ w.r.t. ω, ω' and S .

Theorem 3.4. *Let X be a compact Ptolemy space with properties (E) and (I). Then for every $\omega \in X$ there is a 1-Lipschitz submetry $\pi_\omega : X_\omega \rightarrow B_\omega$ with the base B_ω isometric to an Euclidean space \mathbb{R}^k , $k \geq 1$, such that any Möbius automorphism $\varphi : X \rightarrow X$ with $\varphi(\omega) = \omega'$ induces a homothety $\bar{\varphi} : B_\omega \rightarrow B_{\omega'}$ with $\pi_{\omega'} \circ \varphi = \bar{\varphi} \circ \pi_\omega$.*

The fibers of π_ω , also called \mathbb{K} -lines, have the property

(\mathbb{K}) *given a \mathbb{K} -line $F \subset X_\omega$ and $x \in X_\omega \setminus F$, there is a unique Ptolemy line $l \subset X_\omega$ through x that intersects F .*

Remark 3.5. Recall that a map $f : X \rightarrow Y$ between metric spaces is called a *submetry* if for every ball $B_r(x) \subset X$ of radius $r > 0$ centered at x its image $f(B_r(x))$ coincides with the ball $B_r(f(x)) \subset Y$.

The proof occupies the rest of sect. 3 and sections 4–5. In the remaining parts of this section we give first consequences of the properties (E) and (I). In particular we prove the existence of homotheties (H), the existence of shifts, two point homogeneity and the Busemann flatness of X .

In what follows, we always consider the weak topology on the group $\text{Aut } X$ of Möbius automorphisms of X , i.e. a sequence $\varphi_i \in \text{Aut } X$ converges to $\varphi \in \text{Aut } X$, $\varphi_i \rightarrow \varphi$, if and only if $\varphi_i(x) \rightarrow \varphi(x)$ for every $x \in X$.

3.2 Möbius automorphisms and homothety property (H)

In this section we establish some important additional properties of a Ptolemy space X which follow from (E) and (I).

Given two distinct points $\omega, \omega' \in X$, we denote by $C_{\omega, \omega'}$ the set of all the Ptolemy circles $\sigma \subset X$ through ω, ω' , and by $\Gamma_{\omega, \omega'}$ the group of Möbius automorphisms $\varphi : X \rightarrow X$ such that $\varphi(\omega) = \omega$, $\varphi(\omega') = \omega'$, $\varphi(\sigma) = \sigma$ and φ preserves orientations of σ for every $\sigma \in C_{\omega, \omega'}$.

Proposition 3.6. *Any Ptolemy space X with properties (E) and (I) possesses the following property*

(H) *Homothety: for each distinct $\omega, \omega' \in X$ the group $\Gamma_{\omega, \omega'}$ acts transitively on every arc of $\sigma \setminus \{\omega, \omega'\}$ for every circle $\sigma \in C_{\omega, \omega'}$.*

Remark 3.7. If one of the points ω, ω' is infinitely remote for a metric d of the Möbius structure, then every $\gamma \in \Gamma_{\omega, \omega'}$ is a homothety w.r.t. d . This is why we use (H) for the notation of the property above.

Proof. We assume that ω' is infinitely remote for a metric $d \in \mathcal{M}$. Then for any $\sigma \in C_{\omega, \omega'}$ the curve $\sigma_{\omega'} = \sigma \setminus \omega'$ is a Ptolemy line w.r.t. d , and any $\gamma \in \Gamma_{\omega, \omega'}$ acts on $\sigma_{\omega'}$ as a preserving orientation homothety.

Composing s-inversions $\varphi = \varphi_{\omega, \omega', S}$, $\varphi' = \varphi_{\omega, \omega', S'}$ of X , where $S, S' \subset X$ are spheres between ω, ω' , we obtain a Möbius automorphism $\gamma = \varphi' \circ \varphi$ with properties $\gamma(\omega) = \omega$, $\gamma(\omega') = \omega'$ and $\gamma(\sigma) = \sigma$ for any Ptolemy circle $\sigma \in C_{\omega, \omega'}$. Having no fixed point, both φ, φ' preserve orientations of σ . Hence, γ preserves its orientations, thus γ acts on every arc of $\sigma \setminus \{\omega, \omega'\}$ as a homothety. That is, $\gamma \in \Gamma_{\omega, \omega'}$.

Let $r, r' > 0$ be the radii of S, S' respectively w.r.t. the metric d , $S = S_r^d(\omega)$, $S' = S_{r'}^d(\omega)$. Then for every $x \in X \setminus \{\omega, \omega'\}$ we have $d(\varphi(x), \omega) = \frac{r^2}{d(x, \omega)}$ and $d(\gamma(x), \omega) = d(\varphi' \circ \varphi(x), \varphi'(\omega')) = \frac{r'^2}{d(\varphi(x), \omega)} = (r'/r)^2 d(x, \omega)$. Therefore, the dilatation coefficient of γ equals $\lambda := (r'/r)^2$, and it can be chosen arbitrarily by changing S, S' appropriately. \square

Corollary 3.8. *Any two distinct Ptolemy circles in a Ptolemy space with properties (E) and (I) have in common at most two points.*

Proof. Assume $\omega, \omega', x \in \sigma \cap \sigma'$ are distinct common points of Ptolemy circles $\sigma, \sigma' \subset X$. We have $\gamma(x) \in \sigma \cap \sigma'$ for every $\gamma \in \Gamma_{\omega, \omega'}$. Then by property (H), the arcs of σ and σ' between ω, ω' which contain x coincide. Taking ω'' inside of this common arc and applying the same argument to $\omega', \omega'', x = \omega$, we obtain $\sigma = \sigma'$. \square

3.3 Busemann parallel lines, pure homotheties and shifts

In this section we assume that the compact Ptolemy space X possesses the properties (E) and (I).

We say that Ptolemy lines $l, l' \subset X_\omega$ are *Busemann parallel* if l, l' share Busemann functions, that is, any Busemann function associated with l is also a Busemann function associated with l' and vice versa.

Lemma 3.9. *Let $l, l' \subset X_\omega$ be Ptolemy lines with a common point, $o \in l \cap l'$, $b : X_\omega \rightarrow \mathbb{R}$ a Busemann function of l with $b(o) = 0$. Assume $b \circ c(t) = -t = b \circ c'(t)$ for all $t \geq 0$ and for appropriate unit speed parameterizations $c, c' : \mathbb{R} \rightarrow X_\omega$ of l, l' respectively with $c(0) = o = c'(0)$. Then $l = l'$. In*

particular, Busemann parallel Ptolemy lines coincide if they have a common point.

Proof. We show that the concatenation of $c|(-\infty, 0]$ with $c'|[0, \infty)$ is also a Ptolemy line. Then $l = l'$ by Corollary 3.8. It suffices to show that for $s, t \geq 0$ we have $|c(-s)c'(t)| = t + s$. By triangle inequality we have $|c(-s)c'(t)| \leq t + s$. Letting $t_i \rightarrow \infty$ we have $|c'(t)c(t_i)| - t_i \rightarrow b \circ c'(t) = -t$. Thus by triangle inequality again, we have

$$|c(-s)c'(t)| \geq |c(-s)c(t_i)| - |c'(t)c(t_i)| = (t_i + s) - |c'(t)c(t_i)| \rightarrow t + s.$$

Thus $|c(-s)c'(t)| = t + s$. \square

Next, we show that a sublinear divergence of Ptolemy lines is equivalent for them to be Busemann parallel.

Lemma 3.10. *If two Ptolemy lines $l, l' \subset X_\omega$ are Busemann parallel, then they diverge at most sublinearly, that is $|c(t)c'(t)|/|t| \rightarrow 0$ as $|t| \rightarrow \infty$ for appropriate unit speed parameterizations c, c' of l, l' .*

Conversely, if $|c(t_i)c'(t_i)|/|t_i| \rightarrow 0$ for some sequences $t_i \rightarrow \pm\infty$, then the lines l, l' are Busemann parallel.

Proof. Let $c, c' : \mathbb{R} \rightarrow X_\omega$ be unit speed parameterizations of Busemann parallel lines $l, l' \subset X_\omega$ respectively, and a common Busemann function $b : X_\omega \rightarrow \mathbb{R}$ such that $b \circ c(t) = b \circ c'(t) = -t$ for all $t \in \mathbb{R}$. Let $\mu(t) := |c(t)c'(t)|$. We claim that $\mu(t)/|t| \rightarrow 0$ for $t \rightarrow \pm\infty$. Assume to the contrary, that W.L.G. there exists a sequence $t_i \rightarrow \infty$ with $\mu(t_i)/t_i \geq a > 0$.

By the homothety property (H) there exists a homothety φ_i of X_ω with factor $1/t_i$ such that $\varphi_i \circ c(s) = c(s/t_i)$ for all $s \in \mathbb{R}$. Note that $c'_i(s) = \varphi_i \circ c'(t_i s)$ is a unit speed parameterization of the Ptolemy line $\varphi_i(l')$. For fixed i we calculate

$$\begin{aligned} b \circ c'_i(t) &= \lim_{s \rightarrow \infty} (|c'_i(t)c(s)| - s) = \lim_{s \rightarrow \infty} (|\varphi_i(c'(tt_i))c(s)| - s) \\ &= \lim_{s \rightarrow \infty} (|\varphi_i(c'(tt_i))c(s/t_i)| - s/t_i) = \lim_{s \rightarrow \infty} (|\varphi_i(c'(tt_i))\varphi_i(c(s))| - s/t_i) \\ &= \lim_{s \rightarrow \infty} \frac{1}{t_i} (|c'(tt_i)c(s)| - s) = \frac{1}{t_i} b(c'(tt_i)) = \frac{1}{t_i} (-tt_i) = -t \end{aligned}$$

for all $t \in \mathbb{R}$. The Ptolemy lines $\varphi_i(l')$ subconverge to a Ptolemy line l'' through $c(0)$. If $c'' : \mathbb{R} \rightarrow X$ is the limit unit speed parameterization of l'' , then $b \circ c''(t) = -t$ for all $t \in \mathbb{R}$, and $|c''(1)c(1)| \geq a > 0$. This contradicts Lemma 3.9 by which $l = l'$ and thus $c''(t) = c(t)$ for all $t \in \mathbb{R}$.

Conversely, assume $c, c' : \mathbb{R} \rightarrow X_\omega$ are unit speed parameterizations of Ptolemy lines $l, l' \subset X_\omega$ with $c(0) = o, c'(0) = o'$ such that $b(o) = b(o') = 0$ for the Busemann function $b : X_\omega \rightarrow \mathbb{R}$ of l with $b \circ c(t) = -t, t \in \mathbb{R}$, and $\mu(t_i)/t_i \rightarrow 0$ for some sequence $t_i \rightarrow \infty$, where $\mu(t) = |c(t)c'(t)|$. Let

$b' : X_\omega \rightarrow \mathbb{R}$ be the Busemann function of l' with $b' \circ c'(t) = -t$. Applying the Ptolemy inequality to the cross-ratio triple $\text{crt}(Q_i)$ of the quadruple $Q_i = (o, c(t_i), c'(t_i), o')$, we obtain

$$||oc'(t_i)||o'c(t_i)| - |oc(t_i)||o'c'(t_i)|| \leq |oo'| |c(t_i)c'(t_i)|.$$

Using $|oc(t_i)| = t_i = |o'c'(t_i)|$, $|o'c(t_i)| = b(o') + t_i + o(1)$, $|oc'(t_i)| = b'(o) + t_i + o(1)$, and $|c(t_i)c'(t_i)| = \mu(t_i) = o(1)t_i$, we obtain

$$|(b'(o) + t_i + o(1))(b(o') + t_i + o(1)) - t_i^2| \leq |oo'|o(1)t_i,$$

thus $|b'(o)| \leq o(1)$ and hence $b'(o) = b(o') = 0$.

Finally, for an arbitrary $x \in X_\omega$ consider the quadruple $Q_{x,i} = (x, c(t_i), c'(t_i), o)$. By the same argument as above, we have

$$||xc'(t_i)|t_i - |xc(t_i)||oc'(t_i)|| \leq |ox|\mu(t_i).$$

Using $|oc'(t_i)| = b'(o) + t_i + o(1) = t_i + o(1)$, $|xc'(t_i)| = b'(x) + t_i + o(1)$, $|xc(t_i)| = b(x) + t_i + o(1)$, we finally obtain $|b'(x) - b(x)| \leq o(1)$ and hence $b(x) = b'(x)$. Therefore, the lines l, l' are Busemann parallel. \square

Given $x, x' \in X_\omega$, we construct an isometry $\eta_{xx'} : X_\omega \rightarrow X_\omega$ called a *shift* as follows. We take a sequence $\lambda_i \rightarrow \infty$ and using the homothety property (H) for every i consider homotheties $\varphi_i \in \Gamma_{\omega,x}$, $\psi_i \in \Gamma_{\omega,x'}$ with coefficient λ_i . Then $\eta_i = \psi_i^{-1} \circ \varphi_i$ is an isometry of X_ω for every i because the coefficient of the homothety η_i is 1. Furthermore, we have $|\eta_i(x)x'| = \lambda_i^{-1}|xx'| \rightarrow 0$ as $i \rightarrow \infty$. Since X is compact, the sequence η_i subconverges to an isometry $\eta = \eta_{xx'}$ with $\eta(x) = x'$. The term shift for η is justified by the following

Lemma 3.11. *A shift $\eta_{xx'}$ moves any Ptolemy line l through x to a Busemann parallel Ptolemy line $\eta_{xx'}(l)$ through x' .*

Proof. We show that the line $l' = \eta_{xx'}(l)$ cannot have at least linear divergence with l . Assume to the contrary that $\mu(t) \geq at$ for some $a > 0$ and all $t > 0$, where $\mu(t) = |c(t)c'(t)|$, $c : \mathbb{R} \rightarrow X_\omega$ is a unit speed parameterization of l with $c(0) = x$, $c' = \eta_{xx'} \circ c$.

Recall that $\eta_{xx'} = \lim \eta_i$, where $\eta_i = \psi_i^{-1} \circ \varphi_i$, and $\varphi_i \in \Gamma_{\omega,x}$, $\psi_i \in \Gamma_{\omega,x'}$ are homotheties with the same coefficient $\lambda_i \rightarrow \infty$. By definition of the groups $\Gamma_{\omega,x}$, $\Gamma_{\omega,x'}$, we have $\varphi_i(l) = l$, $\psi_i(l') = l'$. We take $y = c(1)$, $y' = c'(1)$. Then for $y_i = \varphi_i(y) = c(\lambda_i)$ we have $|y_i c'(\lambda_i)| = \mu(\lambda_i) \geq a\lambda_i$. Thus for $y'_i = \psi_i^{-1}(y_i)$ the estimate $|y'_i y'| = |\psi_i^{-1}(y_i)\psi_i^{-1} \circ c'(\lambda_i)| \geq a$ holds for all i in contradiction with $y'_i \rightarrow y'$ as $i \rightarrow \infty$.

Therefore, there are sequences $t_i \rightarrow \pm\infty$, with $\mu(t_i) = o(1)|t_i|$. By Lemma 3.10 the lines l, l' are Busemann parallel. \square

From Lemma 3.9 and Lemma 3.11 we immediately obtain

Corollary 3.12. *Given a Ptolemy line $l \subset X_\omega$, through any point $x \in X_\omega$ there is a unique Ptolemy line $l(x)$ Busemann parallel to l . \square*

Recall that any Möbius map $\varphi : X \rightarrow X$ with $\varphi(\omega) = \omega$ for $\omega \in X$ acts on X_ω as a homothety. A homothety $\varphi : X_\omega \rightarrow X_\omega$ is said to be *pure* if it preserves any foliation of X_ω by Busemann parallel Ptolemy lines.

Lemma 3.13. *For every $o \in X_\omega$ the group $\Gamma_{\omega,o}$ consists of pure homotheties. In particular, every shift of X_ω preserves any foliation of X_ω by Busemann parallel Ptolemy lines.*

Proof. Let $l \subset X_\omega$ be a Ptolemy line through o , $b : X_\omega \rightarrow \mathbb{R}$ a Busemann function of l with $b(o) = 0$. Then $b \circ \varphi = \lambda b$ for every homothety $\varphi \in \Gamma_{\omega,o}$, where $\lambda > 0$ is the coefficient of φ . By Corollary 3.12, any Busemann function of any Ptolemy line $l(x)$ through $x \in X_\omega$ is a Busemann function of a line l through o . Therefore, every $\varphi \in \Gamma_{\omega,o}$ preserves any Busemann function b of $l(x)$ with $b(o) = 0$ in the sense that $\lambda^{-1}b \circ \varphi = b$, where $\lambda > 0$ is the coefficient of φ . Since $\lambda^{-1}b \circ \varphi$ is a Busemann function of the Ptolemy line $\varphi^{-1}(l(x))$, we see that this line is Busemann parallel to $l(x)$. Thus φ preserves the foliation $l(x)$, $x \in X_\omega$ by Busemann parallel Ptolemy lines. \square

A construction of a homothety from the group $\Gamma_{\omega,\omega'}$ given in Proposition 3.6 is not uniquely determined because to obtain a homothety with the same coefficient λ one can take a composition of different pairs of s-inversions. Thus for given $x, x' \in X_\omega$ a shift $\eta_{xx'}$ is not uniquely determined. We give a refined construction of shifts with property $\eta_{xx'} \rightarrow \text{id}$ as $x \rightarrow x'$ which will be used in the proof of Lemma 5.7 below.

Lemma 3.14. *For $x, x' \in X_\omega$ there is a shift $\eta_{xx'} : X_\omega \rightarrow X_\omega$ with $\eta_{xx'}(x) = x'$ such that $\eta_{xx'} \rightarrow \text{id}$ as $x \rightarrow x'$.*

Proof. For $\lambda_i \rightarrow \infty$ we denote by $S = S_1(x)$, $S_i = S_{\lambda_i}(x)$ the metric spheres in X_ω centered at x of radius 1 and λ_i respectively. Similarly we put $S' = S_1(x')$, $S'_i = S_{\lambda_i}(x')$. Then $\varphi_i = \varphi_{x,\omega,S_i} \circ \varphi_{x,\omega,S} \in \Gamma_{\omega,x}$, $\psi_i = \varphi_{x',\omega,S'_i} \circ \varphi_{x',\omega,S'} \in \Gamma_{\omega,x'}$ are homotheties with the same coefficient λ_i^2 , see the proof of Proposition 3.6. The sequence of isometries $\eta_i = \psi_i^{-1} \circ \varphi_i : X_\omega \rightarrow X_\omega$ converges to a shift $\eta : X_\omega \rightarrow X_\omega$ with $\eta(x) = x'$. We have

$$\eta_i = \varphi_{x',\omega,S'} \circ \varphi_{x',\omega,S'_i} \circ \varphi_{x,\omega,S_i} \circ \varphi_{x,\omega,S}$$

and $\varphi_{x',\omega,S'_i} \circ \varphi_{x,\omega,S_i} \rightarrow \text{id}$, $\varphi_{x',\omega,S'} \circ \varphi_{x,\omega,S} \rightarrow \text{id}$ as $x \rightarrow x'$ because $S_i \rightarrow S'_i$, $S \rightarrow S'$ in the Hausdorff metric, every s-inversion is uniquely determined by its data according to our assumption, and s-inversions preserve the family of metric spheres between data points, see sect. 2.3. Thus $\eta_i \rightarrow \text{id}$ for every i as $x \rightarrow x'$. Moreover, the convergence of metric spheres around x to metric spheres around x' in the Hausdorff metric is uniform in radius as $x \rightarrow x'$, $\text{Hd}(S_r^d(x), S_r^d(x')) \leq |xx'|$ for every $r > 0$. Therefore, $\eta_i \rightarrow \text{id}$ uniformly in i as $x \rightarrow x'$. This implies $\eta \rightarrow \text{id}$ as $x \rightarrow x'$. \square

3.4 Two point homogeneity

By property (E) formulated in sect. 1 we know that the space X contains at least one Ptolemy circle.

Proposition 3.15. *Any compact Ptolemy space with the inversion property (I) is two point homogeneous, that is, for each (ordered) pairs (x, y) , (x', y') of distinct points in X there is a Möbius automorphism $f : X \rightarrow X$ with $f(x) = x'$, $f(y) = y'$.*

Proof. Applying an inversion, we can map x to x' . Let y'' be the image of y under the inversion. Then $y'' \neq x'$ by the assumption. We consider a metric of the Möbius structure with infinitely remote point x' . By discussion above, there is a shift w.r.t. that metric which maps y'' to y' . The resulting composition gives a required Möbius automorphism. \square

Remark 3.16. In another terminology, under conditions of Proposition 3.15, the action of the group $\text{Aut } X$ of Möbius transformations on X is *2-transitive*, i.e. $\text{Aut } X$ acts transitively on all pairs $(x, x') \in X \times X$ with distinct entries, see [Kr].

It immediately follows from Proposition 3.15 that the property (E) in any compact Ptolemy space with (I) is promoted to (E) Enhanced existence: through any two points in X there is a Ptolemy circle.

In what follow, we use this property under the name (E).

3.5 Busemann functions and Busemann flatness

The proof of Theorem 3.4 is based on study of Busemann functions on X_ω . In this section we assume that a compact Ptolemy space X possesses the properties (E) and (I).

Lemma 3.17. *Assume that $x_i \rightarrow x$ in X , and a point $\omega \in X$ distinct from x is fixed. Then any Ptolemy circle $l \subset X$ through ω , x is the (pointwise) limit of a sequence of Ptolemy circles $l_i \subset X$ through ω , x_i .*

Proof. In the space X_ω the circle l is a Ptolemy line (with infinitely remote point ω) through x . Then the sequence $l_i = \eta_i(l)$ of Ptolemy lines with $x_i \in l_i$ converges to l , where $\eta_i : X_\omega \rightarrow X_\omega$ is a shift with $\eta_i(x) = x_i$, because the lines l_i are Busemann parallel to l by Lemma 3.13, and any sublimit of the sequence $\{l_i\}$ coincides with l by Lemma 3.9. \square

We fix $\omega \in X$ and a metric d of the Möbius structure such that ω is the infinitely remote point. It immediately follows from the Ptolemy inequality that the distance function $d(z, \cdot)$ to a point $z \in X_\omega$ is convex along any Ptolemy line in X_ω , see [FS2]. Under the homothety property (H) we prove that in fact $d(z, \cdot)$ is C^1 -smooth.

Proposition 3.18. *A compact Ptolemy space with properties (E) and (I) is Busemann flat.*

Proof. By Proposition 2.10 it suffices to prove that the distance function $d_z = d(z, \cdot) : X_\omega \rightarrow \mathbb{R}$ is C^1 -smooth along any Ptolemy line $l \subset X_\omega$ for any $\omega \in X$, $z \in X_\omega \setminus l$.

Assume that d_z is not C^1 -smooth at some point $x \in l$. We fix an arclength parameterization $c : \mathbb{R} \rightarrow X_\omega$ of l such that $x = c(0)$. Since $f = d_z \circ c$ is convex, it has the left and the right derivatives at every point. By assumption, these derivatives are different at $t = 0$. It follows that

$$\liminf_{t \rightarrow 0} \frac{f(t) - 2f(0) + f(-t)}{t} > 0.$$

By Proposition 3.6, property (H) holds for X . Now, using (H), we find for every $\lambda > 0$ a homothety $h_\lambda : X_\omega \rightarrow X_\omega$ with coefficient λ that preserves the point x and the Ptolemy line l , $h_\lambda(x) = x$, $h_\lambda(l) = l$. Then $d(x, h_\lambda(z)) \rightarrow \infty$ as $\lambda \rightarrow \infty$ and thus $\omega_\lambda = h_\lambda(z) \rightarrow \omega$. By Lemma 3.17, there is a Ptolemy circle l_λ through x, ω_λ such that $l_\lambda \rightarrow l$ as $\lambda \rightarrow \infty$. We put $\lambda = 1/t$ and consider points $x_t^+, x_t^- \in l_\lambda$ separated by x with $d(x, x_t^\pm) = 1$. Then, W.L.G., $x_t^\pm \rightarrow c(\pm 1)$ as $t \rightarrow 0$. The points $x_t^+, x, x_t^-, \omega_\lambda$ lie on the Ptolemy circle l_λ (in this order), thus

$$2d(x, \omega_\lambda) \geq d(x, \omega_\lambda)d(x_t^+, x_t^-) = d(x_t^+, \omega_\lambda) + d(x_t^-, \omega_\lambda)$$

by the Ptolemy equality. On the other hand, $f(0)/t = d(x, \omega_\lambda)$ and $f(\pm t)/t = d(c(\pm 1), \omega_\lambda)$. Thus $|d(x_t^\pm, \omega_\lambda) - f(\pm t)/t| \leq d(x_t^\pm, c(\pm 1)) \rightarrow 0$ as $t \rightarrow 0$. Therefore, $(f(t) - 2f(0) + f(-t))/t \rightarrow 0$ as $t \rightarrow 0$ in contradiction with our assumption. \square

Using Proposition 2.9, we immediately obtain

Corollary 3.19. *Given two Ptolemy lines $l, l' \in X_\omega$, the Busemann functions of l are affine functions on l' .* \square

Lemma 3.20. *For any Busemann function $b : X_\omega \rightarrow \mathbb{R}$ of any Ptolemy line $l \subset X_\omega$, every horosphere $H_t = b^{-1}(t)$, $t \in \mathbb{R}$, is geodesically convex, that is, any Ptolemy line $l' \subset X_\omega$ having two distinct points z, z' in common with H_t is contained in H_t , $l' \subset H_t$.*

Proof. We put $b^+ = b$ and assume W.L.G. that $b^+(z) = 0 = b^-(z)$, where the Busemann function b^- of l is opposite to b^+ . Then $b^+ + b^- \equiv 0$ because X is Busemann flat, see Proposition 3.18. Thus $H_0 = (b^+)^{-1}(0) = (b^-)^{-1}(0)$ is a common horosphere for b^+, b^- . Since horoballs, i.e. sublevel sets of Busemann functions, are convex, the geodesic segment $zz' \subset l$ lies in H_0 . By Corollary 3.19, the function b is affine along l' , that is, $b \circ c(t) = \alpha t + \beta$ for any arclength parameterization $c : \mathbb{R} \rightarrow l'$ of l' and some $\alpha, \beta \in \mathbb{R}$, $|\alpha| \leq 1$.

We choose c so that $c(0) = z$, $c(|zz'|) = z'$. Then $\beta = 0$ by the assumption $b(z) = 0$, and $0 = b(z') = b \circ c(|zz'|) = \alpha|zz'|$. Hence $\alpha = 0$ and $b|l' \equiv 0$. This shows that $l' \subset H_0$. \square

4 Slope of Ptolemy lines and circles

4.1 Definition and properties

By Corollary 3.19, a Busemann function associated with a Ptolemy line is affine along any other Ptolemy line. We introduce a quantity which measures a mutual position of Ptolemy lines in the space.

Let $l, l' \subset X_\omega$ be oriented Ptolemy lines. We define the *slope* of l' w.r.t. l as the coefficient of a Busemann function b associated with l when restricted to l' , $\text{slope}(l'; l) = \alpha$ if and only if $b \circ c'(t) = \alpha t + \beta$ for some $\beta \in \mathbb{R}$ and all $t \in \mathbb{R}$, where $c' : \mathbb{R} \rightarrow X_\omega$ is a unit speed parameterization of l' compatible with its orientation. The quantity $\text{slope}(l'; l) \in [-1, 1]$ is well defined, i.e. it depends of the choice neither the Busemann function b nor the parameterization c' (we assume that b is defined via a parameterization of l compatible with its orientation). Note that the slope changes the sign when the orientation of l or l' is changed,

$$\text{slope}(-l'; l) = -\text{slope}(l'; l) = \text{slope}(l'; -l).$$

The first equality is obvious, while the second one holds because X is Busemann flat by Proposition 3.18.

By definition, we have $\text{slope}(l; l) = -1$ for any oriented Ptolemy line $l \subset X_\omega$. More generally, let $l, l' \subset X_\omega$ be Busemann parallel Ptolemy lines. If an orientation of l is fixed, then a *compatible* orientation of l' is well defined. Indeed, we take a Busemann function b of l such that $b \rightarrow -\infty$ along l in the chosen direction. Since b is also a Busemann function of l' , the respective direction of l' such that $b \rightarrow -\infty$ along l' is well defined, and it is independent of the choice of b .

Now, if orientations of Busemann parallel l, l' are compatible, then $\text{slope}(l'; l) = -1 = \text{slope}(l; l')$.

Lemma 4.1. *Let $l, l' \subset X_\omega$ be Busemann parallel Ptolemy lines with compatible orientations. Then for any oriented Ptolemy line $l'' \subset X_\omega$ we have $\text{slope}(l; l'') = \text{slope}(l'; l'')$.*

Proof. Let $b : X_\omega \rightarrow \mathbb{R}$ be a Busemann function associated with l'' . There are unit speed parameterizations $c : \mathbb{R} \rightarrow l$, $c' : \mathbb{R} \rightarrow l'$ compatible with the orientations of l, l' such that $b \circ c(0) = b \circ c'(0) =: \beta$. Then, since b is affine along Ptolemy lines, $b \circ c(t) = \alpha t + \beta$, $b \circ c'(t) = \alpha' t + \beta$ for some $|\alpha|, |\alpha'| \leq 1$ and all $t \in \mathbb{R}$. We show that $\alpha = \alpha'$.

Since the orientations of l, l' are compatible, we have $|c(t)c'(t)| = o(1)|t|$ as $|t| \rightarrow \infty$ by Lemma 3.10. Let $c'' : \mathbb{R} \rightarrow l''$ be a unit speed parameterization such that $b(x) = \lim_{s \rightarrow \infty} |c''(s)x| - s$, $x \in X_\omega$. Since $||c''(s)c(t)| - |c''(s)c'(t)|| \leq |c(t)c'(t)| = o(1)t$ as $t \rightarrow \infty$, we have

$$|(\alpha - \alpha')t| = |b \circ c(t) - b \circ c'(t)| = o(1)t$$

and hence $\alpha = \alpha'$. \square

Proposition 2.9 combined with duality gives rise to a first variation formula to describe which we use the following agreement. Let $\sigma, \sigma' \subset X$ be Ptolemy circles meeting each other at two distinct points ω and ω' , $\sigma \cap \sigma' = \{\omega, \omega'\}$, which decompose the circles into (closed) arcs $\sigma = \sigma_+ \cup \sigma_-$ and $\sigma' = \sigma'_+ \cup \sigma'_-$. The choice of ω as an infinitely remote point automatically introduces the orientation of σ as well as of σ' such that ω' is the initial point of the arcs σ_+, σ'_+ , while ω the final point of σ_+, σ'_+ , and the similar agreement holds true for the choice of ω' as an infinitely remote point. Then the slope($\sigma'_\omega; \sigma_\omega$) of the Ptolemy line $\sigma'_\omega \subset X_\omega$ w.r.t. the Ptolemy line $\sigma_\omega \subset X_\omega$ is well defined.

Lemma 4.2. *Let $\sigma, \sigma' \subset X$ be Ptolemy circles meeting each other at two distinct points ω and ω' , $\sigma \cap \sigma' = \{\omega, \omega'\}$, which decompose the circles into (closed) arcs $\sigma = \sigma_+ \cup \sigma_-$ and $\sigma' = \sigma'_+ \cup \sigma'_-$. Let $c : \mathbb{R} \rightarrow X_{\omega'}$, $c' : \mathbb{R} \rightarrow X_\omega$ be the unit speed parameterizations of the oriented Ptolemy lines $\sigma_{\omega'} \subset X_{\omega'}$, $\sigma'_\omega \subset X_\omega$ respectively compatible with the orientations such that $c(0) = \omega$, $c'(0) = \omega'$. Then*

$$\frac{d}{dt} d'(c'(s), c(t))|_{t=0} = \alpha \operatorname{sign} s \quad (7)$$

for all $s \neq 0$, where d' is the metric of $X_{\omega'}$, and $\alpha = \operatorname{slope}(\sigma'_\omega; \sigma_\omega)$.

Remark 4.3. We emphasize that (7) is a typical duality equality where the left hand side is computed in the space $X_{\omega'}$, while the right hand side is computed in the inverted space X_ω .

Proof. By Corollary 3.19, the Busemann function b of σ_ω with $b(\omega') = 0$ is affine along the Ptolemy line $\sigma'_\omega \subset X_\omega$. Thus $b \circ c'(s) = \alpha s$ for $\alpha = \operatorname{slope}(\sigma'_\omega; \sigma_\omega)$ and all $s \in \mathbb{R}$ because $b \circ c'(0) = b(\omega') = 0$. Since X is Busemann flat by Proposition 3.18, Equation (5) applied to $b^+ = b$ gives

$$\alpha s = b \circ c'(s) = \frac{d}{dt} \ln d'(c'(s), c(t))|_{t=0} = \frac{1}{d'(c'(s), \omega)} \frac{d}{dt} d'(c'(s), c(t))|_{t=0}$$

for all $s \neq 0$. Using that $d'(c'(s), \omega) = 1/|s|$, we obtain the required equality. \square

In the situation with two Ptolemy circles intersecting at two distinct points as in Lemma 4.2, we have four a priori different slopes. The duality and existence of s-inversions allows to reduce this number to one.

Lemma 4.4. *Let $\sigma, \sigma' \subset X$ be Ptolemy circles meeting each other at two distinct points ω and ω' , $\sigma \cap \sigma' = \{\omega, \omega'\}$, which decompose the circles into (closed) arcs $\sigma = \sigma_+ \cup \sigma_-$ and $\sigma' = \sigma'_+ \cup \sigma'_-$. Then*

$$\text{slope}(\sigma'_\omega; \sigma_\omega) = \text{slope}(\sigma_{\omega'}; \sigma'_{\omega'}).$$

Proof. Denote $\alpha = \text{slope}(\sigma'_\omega; \sigma_\omega)$ and $\alpha' = \text{slope}(\sigma_{\omega'}; \sigma'_{\omega'})$. As in Lemma 4.2 let $c : \mathbb{R} \rightarrow X_{\omega'}$, $c' : \mathbb{R} \rightarrow X_\omega$ be the unit speed parameterizations of the oriented Ptolemy lines $\sigma_{\omega'} \subset X_{\omega'}$, $\sigma'_\omega \subset X_\omega$ respectively compatible with the orientations such that $c(0) = \omega$, $c'(0) = \omega'$. Let $b' : X_{\omega'} \rightarrow \mathbb{R}$ be the Busemann function associated with $\sigma'_{\omega'}$ such that $b'(\omega) = 0$ (according to our agreement, b' is computed via a parameterization of $\sigma'_{\omega'}$ which is opposite in orientation to that of the parameterization c'). Then $b' \circ c(t) = \alpha' t$ for all $t \in \mathbb{R}$ by the definition of $\alpha' = \text{slope}(\sigma_{\omega'}; \sigma'_{\omega'})$. Thus

$$\alpha' t = b' \circ c(t) \leq d'(c'(s), c(t)) - d'(c'(s), \omega)$$

for all $s > 0$ and all (sufficiently small) $t \in \mathbb{R}$, where d' is the metric of $X_{\omega'}$. The last inequality holds because the right hand side decreases monotonically to $b' \circ c(t)$ for every fixed t as $s \rightarrow 0$. Applying Equality (7), we obtain $\alpha' \leq \alpha$. Interchanging ω with ω' and σ with σ' , we have $\alpha \leq \alpha'$ by the same argument. Hence, the claim. \square

Using Lemma 4.4, the first variation formula (7) can be rewritten as follows

$$\frac{d}{dt} d'(c'(s), c(t))|_{t=0} = \alpha' \text{sign } s \quad (8)$$

for all $s \neq 0$, where $\alpha' = \text{slope}(\sigma_{\omega'}; \sigma'_{\omega'})$. Now, the both sides of (8) are computed in the same space $X_{\omega'}$.

Lemma 4.4 implies the symmetry of the slope w.r.t. the arguments.

Lemma 4.5. *For any oriented Ptolemy lines $l, l' \subset X_\omega$ we have $\text{slope}(l'; l) = \text{slope}(l; l')$.*

Proof. We assume W.L.G. that $l \cap l' = \omega'$ and represent $l = \sigma_\omega$, $l' = \sigma'_\omega$ for Ptolemy circles $\sigma = l \cup \omega$, $\sigma' = l' \cup \omega$. Then $\sigma \cap \sigma' = \{\omega, \omega'\}$. Let $S \subset X$ be a sphere between ω, ω' , $\varphi = \varphi_{\omega, \omega', S} : X \rightarrow X$ the s-inversion w.r.t. ω, ω', S . Then φ preserves any Ptolemy circle through ω, ω' and its orientations. In particular, $\varphi(\sigma_\omega) = \sigma_{\omega'}$ and $\varphi(\sigma'_\omega) = \sigma'_{\omega'}$. We assume that $S = S_1^d(\omega')$, where $d \in \mathcal{M}$ is the metric of X_ω . Then the metric $d' = \varphi^* d$ is the m-inversion of d , and vice versa, see Lemma 3.3. It follows that the map $\varphi : (X_\omega, d) \rightarrow (X_{\omega'}, d')$ is an isometry.

Now, we have

$$\text{slope}(\sigma'_{\omega'}; \sigma_{\omega'}) = \text{slope}(\varphi(\sigma'_\omega); \varphi(\sigma_\omega)) = \text{slope}(\sigma'_\omega; \sigma_\omega) = \text{slope}(l'; l).$$

Using Lemma 4.4 we obtain $\text{slope}(l; l') = \text{slope}(\sigma_\omega; \sigma'_\omega) = \text{slope}(\sigma'_{\omega'}; \sigma_{\omega'}) = \text{slope}(l'; l)$. \square

From now on, we use notation $\text{slope}(l, l')$ for the slope instead of $\text{slope}(l; l')$. We say that Ptolemy lines $l, l' \subset X_\omega$ are *orthogonal* if $\text{slope}(l', l) = 0$. By Lemma 4.5 this is a symmetric relation. For orthogonal lines we also use notation $l \perp l'$.

4.2 Tangent lines and slope of Ptolemy circles

A Ptolemy line $l \subset X_\omega$ is *tangent* to a Ptolemy circle $\sigma \subset X_\omega$ at a point $x \in \sigma$ if for every $y \in \sigma$ sufficiently close to x we have $\text{dist}(y, l) = o(|xy|)$.

Proposition 4.6. *Every Ptolemy circle $\sigma \subset X_\omega$ possesses a unique tangent Ptolemy line l at every point $x \in \sigma$.*

Proof. Let d be the metric of X_ω , d' the metric inversion of d w.r.t. x , $d'(y, z) = \frac{d(y, z)}{d(y, x)d(z, x)}$. Then x is infinitely remote for d' , and $\sigma \setminus x \subset X_x$ is a Ptolemy line w.r.t. the metric d' on X_x . By Corollary 3.12, there is a unique Ptolemy line $\tilde{l} \subset X_x$ through ω which is Busemann parallel to the line $\sigma \setminus x$. Then $l = (\tilde{l} \cup x) \setminus \omega \subset X_\omega$ is a Ptolemy line through x . We show that l is tangent to σ at x .

We fix on $\sigma \setminus x$ and \tilde{l} compatible orientations, see sect. 4.1, and choose $y \in \sigma$, $y' \in l$ with sufficiently small positive $t = d(x, y) = d(x, y')$ according to the orientations. Recall that d is also the metric inversion of d' w.r.t. ω , and that $d(x, z) = 1/d'(\omega, z)$ for every $z \in X \setminus \{x, \omega\}$. Then

$$d(y, y') = \frac{d'(y, y')}{d'(\omega, y)d'(\omega, y')} = t \frac{d'(y, y')}{1/t}.$$

By Lemma 3.10, $\frac{d'(y, y')}{1/t} \rightarrow 0$ as $t \rightarrow 0$, hence $d(y, y') = o(t)$, and thus l is tangent to σ at x .

If $l' \subset X_\omega$ is another tangent line to σ at x , then reversing the argument above we observe that the Ptolemy lines $\tilde{l}, \tilde{l}' = (l' \cup \omega) \setminus x \subset X_x$ through ω diverge sublinearly and thus they are Busemann parallel again by Lemma 3.10. It follows that $\tilde{l} = \tilde{l}'$ and $l = l'$. \square

Now, we reformulate Corollary 3.12 as follows.

Corollary 4.7. *Given a Ptolemy line $l \subset X_\omega$ and a point $x \in l$, for any other point $y \in X_\omega$ there exists a unique Ptolemy circle $\sigma \subset X_\omega$ through y tangent to l at x . In particular, if $y \in l$, then $\sigma = l$.*

Proof. Consider a metric of the Möbius structure on X with the infinitely remote point x and apply Corollary 3.12. \square

If oriented Ptolemy circles $\sigma, \sigma' \subset X$ are disjoint, then their slope is not determined. Assume now that $\omega \in \sigma \cap \sigma'$. Then $\text{slope}_\omega(\sigma, \sigma') \in [-1, 1]$ is defined as the slope of oriented Ptolemy lines $\sigma \setminus \omega, \sigma' \setminus \omega \subset X_\omega$. This is well

defined and symmetric, $\text{slope}_\omega(\sigma, \sigma') = \text{slope}_\omega(\sigma', \sigma)$, by Lemma 4.5. In the case $\text{slope}_\omega(\sigma, \sigma') = \pm 1$, the Ptolemy circles are tangent to each other at ω , having compatible (-1) or opposite $(+1)$ orientations. This means that in any space $X_{\omega'}$ with $\omega' \neq \omega$, the Ptolemy circles $\sigma \setminus \omega'$, $\sigma' \setminus \omega' \subset X_\omega$ have a common tangent line at ω .

More generally, let $l, l' \subset X_{\omega'}$ be the tangent lines to σ, σ' respectively at ω , oriented according to the orientations of σ, σ' . Then $\text{slope}_\omega(\sigma, \sigma') = \text{slope}(l, l')$ because in the space X_ω the Ptolemy line $\sigma \setminus \omega$ is Busemann parallel to $l \setminus \omega$, and $\sigma' \setminus \omega$ is Busemann parallel to $l' \setminus \omega$, and we can apply Lemma 4.1.

Furthermore, if σ, σ' have two distinct common points, $\omega, \omega' \in \sigma \cap \sigma'$, then $\text{slope}_\omega(\sigma, \sigma') = \text{slope}_{\omega'}(\sigma, \sigma')$. This follows from Lemma 4.4 and Lemma 4.5. Thus we use notation $\text{slope}(\sigma, \sigma')$ for the slope of intersecting oriented Ptolemy circles σ, σ' .

5 Fibration $\pi_\omega : X_\omega \rightarrow B_\omega$

5.1 Definition and properties

As usual we fix $\omega \in X$ and consider a metric d of the Möbius structure with infinitely remote point ω .

Given $x \in X_\omega$, we define

$$F_x = \bigcap_{l \ni x} H_l,$$

where the intersection is taken over all the Ptolemy lines $l \subset X_\omega$ through x , H_l is the horosphere through x of a Busemann function associated with l (since X is Busemann flat, H_l is independent of choice of a Busemann function).

Lemma 5.1. *For any $y \in F_x$ we have $F_y = F_x$.*

Proof. By Corollary 3.12, for every Ptolemy line l through x there is a unique Ptolemy line l' through y such that l, l' are Busemann parallel. Let b be a Busemann function of l such that $b(x) = 0$. Then $b(y) = 0$ because y lies in the horosphere through x of b . Hence $H_l = H_{l'}$ because b is a Busemann function also of l' , and thus $F_y = F_x$. \square

By Lemma 5.1, the sets $F_x, F_{x'}$ coincide or are disjoint for any $x, x' \in X_\omega$. We let $B_\omega = \{F_x : x \in X_\omega\}$ and define $\pi_\omega : X_\omega \rightarrow B_\omega$ by $\pi_\omega(x) = F_x$. Therefore, the fibers $F_b = \pi_\omega^{-1}(b)$, $b \in B_\omega$, form a partition of X_ω , B_ω is the factor-space of this partition, and π_ω is the respective factor-map. A fiber F of π_ω is also called a \mathbb{K} -line.

Lemma 5.2. *For any $\omega, \omega' \in X$, any Möbius automorphism $\varphi : X \rightarrow X$ with $\varphi(\omega) = \omega'$ induces a bijection $\bar{\varphi} : B_\omega \rightarrow B_{\omega'}$ such that $\pi_{\omega'} \circ \varphi = \bar{\varphi} \circ \pi_\omega$.*

Proof. It follows from Lemma 2.1 that for any metrics d, d' of the Möbius structure with infinitely remote points ω, ω' respectively, the map $\varphi : (X_\omega, d) \rightarrow (X_{\omega'}, d')$ is a homothety. Thus φ maps any Ptolemy line $l \subset X_\omega$ to the Ptolemy line $l' = \varphi(l) \subset X_{\omega'}$, and $b \circ \varphi^{-1}$ is proportional to a Busemann function of l' for any Busemann function b of l . It follows that φ induces a bijection $\bar{\varphi} : B_\omega \rightarrow B_{\omega'}$ such that $\bar{\varphi} \circ \pi_\omega = \pi_{\omega'} \circ \varphi$. \square

Proof of property (K): uniqueness. Let $F \subset X_\omega$ be a \mathbb{K} -line, and let $x \in X \setminus F$. We show that there is at most one Ptolemy line in X_ω through x that meets F . Assume that there are Ptolemy lines $l, l' \subset X_\omega$ through x that intersect F . Let $c : \mathbb{R} \rightarrow l, c' : \mathbb{R} \rightarrow l'$ be unit speed parameterizations such that $c(0) = x = c'(0)$, and $c(s) \in F, c'(s') \in F$ for some $s, s' > 0$. For the Busemann function $b : X_\omega \rightarrow \mathbb{R}, b(y) = \lim_{t \rightarrow -\infty} |yc(t)| - |t|$, of l we have $b(c(s)) = s = b(c'(s'))$ because $c(s), c'(s') \in F$ and F is a fiber of the fibration $\pi_\omega : X_\omega \rightarrow B_\omega$. The function $t \mapsto |c'(s')c(t)| - (|t| + s)$ is nonincreasing and it converges to $b(c'(s')) - s = 0$ as $t \rightarrow -\infty$, thus $s' = |c'(s')c(0)| \geq s = |c(s)x|$. Interchanging l and l' we obtain $s \geq s'$ by the same reason. Hence $s = s'$. Since the Busemann function b is affine along l' by Corollary 3.19 and it takes the equal values $0 = b \circ c(0)$ and $b \circ c(s) = s = b \circ c'(s)$ along l, l' at two different parameter points, we have $b \circ c(t) = b \circ c'(t)$ for every $t \in \mathbb{R}$. By Lemma 3.9, $l = l'$. \square

Lemma 5.3. *Given a Ptolemy line $l \subset X_\omega$ and distinct points $x, y \in l$, a Ptolemy line $l' \subset X_\omega$ through $x' \in F_x$ meets the fiber F_y if and only if it is Busemann parallel to l . In this case $|x'y'| = |xy|$ for $y' = l' \cap F_y$.*

Proof. By Corollary 3.12, there is a unique Ptolemy line \tilde{l} through x' which is Busemann parallel to l . Consider compatible unit speed parameterizations $c : \mathbb{R} \rightarrow l, \tilde{c} : \mathbb{R} \rightarrow \tilde{l}$ such that $c(0) = x, \tilde{c}(0) = x'$. Then $y = c(t)$ for some $t \in \mathbb{R}$. We show that $\tilde{c}(t) \in F_y$. Let l'' be a Ptolemy line through $c(t)$, b'' a Busemann function of l'' with $b''(c(t)) = 0$. We show that $\tilde{c}(t)$ lies in the zero level set of b'' , $b''(\tilde{c}(t)) = 0$.

By Corollary 3.12, there is a Ptolemy line through $c(0)$ for which b'' is a Busemann function. Then the \mathbb{K} -line F_x lies in a level set of b'' , in particular, $b''(c(0)) = b''(\tilde{c}(0)) =: \beta$. By Lemma 4.1 we have $b'' \circ c(s) = \alpha s + \beta = b'' \circ \tilde{c}(s)$ for all $s \in \mathbb{R}$. In particular, $b''(\tilde{c}(t)) = b''(c(t)) = 0$, hence $\tilde{c}(t) \in F_y$ and \tilde{l} meets F_y . By the uniqueness part of property (K), there is at most one Ptolemy line through x' that hits F_y . Hence, if l' meets F_y , then $l' = \tilde{l}$ is Busemann parallel to l . Moreover, the argument above also shows that the \mathbb{K} -lines F_x, F_y are equidistant, $|x'y'| = |xy|$. \square

5.2 Zigzag curves

We fix $\omega \in X$ and consider a metric on X_ω with infinitely remote point ω . Let $l \subset X_\omega$ be an oriented Ptolemy line. By Corollary 3.12, there is a

foliation $l(x)$, $x \in X_\omega$ of the space X_ω by Ptolemy lines, which are Busemann parallel to l . Moreover, every member $l(x)$ of the foliation has a well defined orientation compatible with that of l , see sect. 4.1.

Lemma 5.4. *Let $l_1, l_2 \subset X_\omega$ be oriented Ptolemy lines which induce respective foliations of X_ω . We start moving from $x \in X_\omega$ along $l_1(x)$ by some distance $s_1 \geq 0$ up to a point y , and then switch to $l_2(y)$ and move along it by some distance $s_2 \geq 0$ up to a point z . Next, we move from $x' \in X_\omega$ along $l_2(x')$ by the distance s_2 up to a point y' , and then switch to $l_1(y')$ and move along it by the distance s_1 up to a point z' , where we always move in the directions prescribed by the orientations. If x, x' lie in a \mathbb{K} -line $F \subset X_\omega$, then z, z' also lie in one and the same \mathbb{K} -line $F' \subset X_\omega$.*

Proof. Let $c_1, c_2 : \mathbb{R} \rightarrow X_\omega$ be the unit speed parameterizations of $l_1(x), l_2(x')$ respectively compatible with the orientations such that $c_1(0) = x, c_2(0) = x'$. We also consider the unit speed parameterizations $c'_1, c'_2 : \mathbb{R} \rightarrow X_\omega$ of $l_1(y'), l_2(y)$ respectively compatible with the orientations such that $c'_1(0) = y', c'_2(0) = y$. Then $c_1(s_1) = y = c'_2(0), c_2(s_2) = y' = c'_1(0)$ and $c'_2(s_2) = z, c'_1(s_1) = z'$.

Let b be a Busemann function of a Ptolemy line $l \subset X_\omega$ which vanishes along F , in particular, $b(x) = 0 = b(x')$. By Corollary 3.19, b is affine along any Ptolemy line in X_ω , in particular, $b \circ c_1(t) = \alpha_1 t, b \circ c_2(t) = \alpha_2 t$ for some α_i which by Lemma 4.1 only depends on $l_i, i = 1, 2$, and for all $t \in \mathbb{R}$. Thus we have $b(z') = b \circ c'_1(s_1) = \alpha_1 s_1 + \alpha_2 s_2$ and similarly $b(z) = b \circ c'_2(s_2) = \alpha_2 s_2 + \alpha_1 s_1$. Hence any Busemann function on X_ω takes the same value at the points z and z' , i.e. these points lie in a common \mathbb{K} -line F' . \square

Given a base point $o \in X_\omega$, a finite ordered collection $\mathcal{L} = \{l_1, \dots, l_k\}$ of oriented Ptolemy lines in X_ω , and a collection $S = \{s_1, \dots, s_k\}$ of non-negative numbers with $s_1 + \dots + s_k > 0$, we construct a sequence $\gamma_p = \gamma_p(o, \mathcal{L}, S) \subset X_\omega, p \geq 1$, of piecewise geodesic curves through o as follows. Recall that we have k foliations of X_ω by oriented Ptolemy lines $l_1(x), \dots, l_k(x), x \in X_\omega$, which are Busemann parallel with compatible orientations to l_1, \dots, l_k respectively.

The curve γ_p starts at $o = v_p^0$ for every $p \geq 1$. We move along $l_1(o)$ by the distance $s_1/2^{p-1}$ up to the point $v_p^1 \in l_1(v_p^0)$, then switch to the line $l_2(v_p^1)$ and move along it by the distance $s_2/2^{p-1}$ up to the point v_p^2 etc. On the i th step, for $1 \leq i \leq k$, we move along the line $l_i(v_p^{i-1})$ by the distance $s_i/2^{p-1}$ in the direction prescribed by the orientation of the line up to the point $v_p^i \in l_i(v_p^{i-1})$. Starting with the point v_p^k we then repeat this procedure only taking the subindices for l_i, s_i modulo k for all integer $i \geq k+1$.

This produces the sequence v_p^n of vertices of γ_p for all $n \geq 0$. For integer $n < 0$ the vertices v_p^n are determined in the same way with all the orientations of the lines l_1, \dots, l_k reversed, with the starting line $l_k(o)$, and

with the ordered collections $\overline{\mathcal{L}} = \{l_k, \dots, l_1\}$ of lines, and $\overline{S} = \{s_k, \dots, s_1\}$ of numbers.

Every curve γ_p receives the arclength parameterization, for which we use the same notation $\gamma_p : \mathbb{R} \rightarrow X_\omega$, with $\gamma_p(0) = o$. Then for every $m \in \mathbb{Z}$, $1 \leq i \leq k$, we have $\gamma_p(t_p^n) = v_p^n$ is a vertex of γ_p , where $n = k(m-1) + i$, $t_p^n = [(s_1 + \dots + s_i)m + (s_{i+1} + \dots + s_k)(m-1)]/2^{p-1}$ (the sum $(s_{i+1} + \dots + s_k)$ is assumed to be zero for $i = k$).

It follows from Lemma 5.4 by induction that for every $n = km \in \mathbb{Z}$, the vertices $v_p^n = \gamma_p(t_p^n)$ of γ_p and $v_{p+1}^{2n} = \gamma_{p+1}(t_{p+1}^{2n})$ of γ_{p+1} lie in a common \mathbb{K} -line in X_ω for every $p \geq 1$. From this one easily concludes that the sequence of the projected curves $\pi_\omega(\gamma_p) \subset B_\omega$ converges (pointwise in the induced topology). At this stage, we do not have tools to prove that the sequence γ_p itself converges in X_ω . However, we need a limiting object of γ_p . Thus, for instance, we fix a nonprincipal ultra-filter on \mathbb{Z} and say that $\gamma = \lim \gamma_p$ w.r.t. that ultra-filter. By this we mean that $\gamma(t) = \lim \gamma_p(t)$ for every $t \in \mathbb{R}$. The curve $\gamma = \gamma(o, \mathcal{L}, S)$ is called a *zigzag curve*, and it is obtained together with the limiting parameterization $\gamma : \mathbb{R} \rightarrow X_\omega$, $\gamma(0) = o$, which in general is not an arclength parameterization.

Lemma 5.5. *Every Busemann function $b : X_\omega \rightarrow \mathbb{R}$ is affine along any zigzag curve γ , that is, the function $b \circ \gamma : \mathbb{R} \rightarrow \mathbb{R}$ is affine. Moreover, if $\gamma = \gamma(o, \mathcal{L}, S)$ for a base point $o \in X_\omega$, some ordered collection $\mathcal{L} = \{l_1, \dots, l_k\}$ of oriented Ptolemy lines in X_ω , and a collection $S = \{s_1, \dots, s_k\}$ of nonnegative numbers with $s_1 + \dots + s_k > 0$, and $b(o) = 0$, then $b \circ \gamma(t) = \beta t$ for all $t \in \mathbb{R}$, where $\beta = \sum_i \alpha_i s_i / \sum_i s_i$, $\alpha_i = \text{slope}(l_i, l)$, $i = 1, \dots, k$, and $l \subset X_\omega$ is the oriented Ptolemy line for which the function b is associated.*

Proof. We assume that $\gamma = \lim \gamma_p$. Since γ_p is piecewise geodesic for every $p \geq 1$, the function $b \circ \gamma_p : \mathbb{R} \rightarrow \mathbb{R}$ is piecewise affine. Recall that the points $v_p^n = \gamma_p(t_p^n)$ are vertices of γ_p , where $t_p^n = [(s_1 + \dots + s_i)m + (s_{i+1} + \dots + s_k)(m-1)]/2^{p-1}$ for $n = k(m-1) + i \in \mathbb{Z}$. Thus we have by induction

$$b \circ \gamma_p(t_p^n) = [(\alpha_1 s_1 + \dots + \alpha_i s_i)m + (\alpha_{i+1} s_{i+1} + \dots + \alpha_k s_k)(m-1)]/2^{p-1}$$

for $n = k(m-1) + i \in \mathbb{Z}$. Hence, $b \circ \gamma_p(t_p^n) = \beta t_p^n + o(1)$ as $p \rightarrow \infty$. Since the step $t_p^{n+1} - t_p^n \leq \max_i s_i / 2^{p-1} \rightarrow 0$ as $p \rightarrow \infty$, we conclude that $b \circ \gamma_p \rightarrow b \circ \gamma$ pointwise as $p \rightarrow \infty$, and $b \circ \gamma(t) = \beta t$ for all $t \in \mathbb{R}$. \square

Lemma 5.5 gives a strong evidence in support of the expectation that a zigzag curve under natural assumptions actually is a Ptolemy line. However we need additional arguments for the proof of this.

For ω , $o \in X$, the group $\Gamma_{\omega, o}$ consists of homotheties $\varphi : X_\omega \rightarrow X_\omega$ with $\varphi(o) = o$ such that $\varphi(l) = l$ for every Ptolemy line $l \subset X_\omega$ through o preserving an orientation of l , and moreover by property (H), $\Gamma_{\omega, o}$ acts transitively on the open rays of l with the vertex o , see Proposition 3.6.

Lemma 5.6. *For any base point $o \in X_\omega$, the homothety $\varphi \in \Gamma_{\omega,o}$ with the coefficient $\lambda = 1/2$ leaves invariant a zigzag curve $\gamma = \gamma(o, \mathcal{L}, S)$ for any ordered collection of oriented Ptolemy lines $\mathcal{L} = \{l_1, \dots, l_k\}$ in X_ω , and any collection $S = \{s_1, \dots, s_k\}$ of nonnegative numbers with $s_1 + \dots + s_k > 0$.*

Proof. Let $\gamma_p = \gamma_p(o, \mathcal{L}, S)$, $p \geq 1$, be the sequence of piecewise geodesic curves in X_ω , so $\gamma = \lim \gamma_p$. For $p \geq 1$, we let v_p^{km} , $m \in \mathbb{Z}$, be the sequence of vertices of γ_p , $\bar{v}_p^{km} = \pi_\omega(v_p^{km})$ the sequence of respective fibers of the fibration $\pi_\omega : X_\omega \rightarrow B_\omega$. Recall that the sequences $\{\bar{v}_p^{km} : m \in \mathbb{Z}\}$, $p \geq 1$, approximate the projection $\pi_\omega(\gamma)$ of γ , that is, $\pi_\omega(\gamma)$ coincides with the closure of the union $\cup_p \{\bar{v}_p^{km} : m \in \mathbb{Z}\}$.

We have $\varphi(\gamma_p) = \gamma_{p+1}$ and $\varphi(v_p^{km}) = v_{p+1}^{km}$ by the construction of γ_p and Lemma 3.13. For every dyadic number $q = m/2^r$, $m \in \mathbb{Z}$, $r \geq 0$, the sequence $v(q) = \{v_p^{q_p} : p \geq r+1\}$, where $q_p = 2^{p-(r+1)} \cdot km$, lies in a common fiber $F = F(q)$ of π_ω . Thus φ maps this sequence into the sequence $v'(q) = \{v_{p+1}^{q'_p} : p \geq r+1\} \subset \varphi(F) = F(q/2)$, where $q'_p = 2^{p-r}(km/2)$, shrinking the mutual distances by the factor $1/2$. Hence for the limit point $x = \lim v(q) \in \gamma$ of any limiting procedure giving $\gamma = \lim \gamma_p$ we have $\varphi(x) = \lim v'(q) \in \gamma$. The points of type $x = \lim v(q)$ with dyadic q are dense in γ , thus φ preserves γ , $\varphi(\gamma) = \gamma$. \square

Lemma 5.7. *Let $\gamma = \gamma(o, \mathcal{L}, S) \subset X_\omega$ be a zigzag curve with base point $o \in X_\omega$, where $\mathcal{L} = \{l_1, \dots, l_k\}$, $S = \{s_1, \dots, s_k\}$, $s_1 + \dots + s_k > 0$. Then for any $o' \in \gamma$ we have $\gamma(o', \mathcal{L}, S) = \gamma$, i.e. any zigzag curve γ is independent of a choice of its base point o .*

Proof. We first consider the case $o' = \gamma(t_q)$ is a *dyadic* point with dyadic $q = m/2^r$, $m \in \mathbb{Z}$, $r \geq 0$, and $t_q = (s_1 + \dots + s_k)q$, for the canonical parameterization $t \mapsto \gamma(t)$ of γ . Then o' is an accumulation point of the vertices $v_p = v_p^{q_p} = \gamma_p(t_p^{q_p})$, $p \geq r+1$, where $q_p = 2^{p-(r+1)} \cdot km$ and $t_p^{q_p} = (s_1 + \dots + s_k)(q_p/k)/2^{p-1} = t_q$, of approximating piecewise geodesic curves γ_p , $\gamma = \lim \gamma_p$ (recall that the sequence $v(q) = \{v_p^{q_p} : p \geq r+1\}$ lies in a fiber $F(q) \subset X_\omega$ of the projection π_ω , see the proof of Lemma 5.6). That is, $o' = \lim v_p$ for our limiting procedure.

By Lemma 3.14, there is a shift $\eta_p = \eta_{v_p o'} : X_\omega \rightarrow X_\omega$ with $\eta_p(v_p) = o'$ and $\lim \eta_p = \text{id}$. Then $\eta_p(\gamma_p) = \gamma'_p$, where $\gamma'_p = \gamma_p(o', \mathcal{L}, S)$ is the piecewise geodesic curve with the base point o' approximating the zigzag curve $\gamma' = \gamma(o', \mathcal{L}, S)$, $\gamma' = \lim \gamma'_p$. Now for an arbitrary point $x \in \gamma$, $x = \gamma(t)$, we have $x = \lim \gamma_p(t)$. We put $x' = \gamma'(t) = \lim \gamma'_p(t)$. Then for an arbitrary $\varepsilon > 0$ we have $|x\gamma_p(t)| < \varepsilon$, $|x'\gamma'_p(t)| < \varepsilon$, and $|x\eta_p(x)| < \varepsilon$ for sufficiently large p . The last estimate holds since $\lim \eta_p = \text{id}$. Using $|\eta_p(x)\gamma'_p(t)| = |\eta_p(x)\eta_p \circ \gamma_p(t)| = |x\gamma_p(t)|$, we obtain

$$|xx'| \leq |x\eta_p(x)| + |\eta_p(x)\gamma'_p(t)| + |\gamma'_p(t)x'| \leq 3\varepsilon,$$

thus $x = x'$, that is, $\gamma = \gamma'$.

For a general case, the point $o' = \gamma(t)$ can be approximated by dyadic ones, $t_q \rightarrow t$. Then respective piecewise geodesic curves $\gamma'_{p,q}$ with dyadic base points $\gamma(t_q)$ approximate pointwise the curve γ'_p with the base point o' for every $p \geq 1$. Thus $\gamma' = \gamma$ also in that case. \square

Proposition 5.8. *Every zigzag curve $\gamma \subset X_\omega$ is either a geodesic and hence a Ptolemy line, or it degenerates to a point. More precisely, if $\gamma = \gamma(o, \mathcal{L}, \mathcal{S})$ for some base point $o \in X_\omega$ and collections $\mathcal{L} = \{l_1, \dots, l_k\}$ of oriented Ptolemy lines in X_ω , $\mathcal{S} = \{s_1, \dots, s_k\}$ of nonnegative numbers with $s_1 + \dots + s_k > 0$, then γ is degenerate if and only if $\sum_i \alpha_i s_i = 0$ for every oriented Ptolemy line $l \subset X_\omega$, where $\alpha_i = \text{slope}(l_i, l)$, $i = 1, \dots, k$.*

Proof. We first show that for each $o, o' \in \gamma$ there is a midpoint $x \in \gamma$. By Lemma 5.7 and Lemma 5.6, homotheties $\varphi \in \Gamma_{\omega, o}$, $\varphi' \in \Gamma_{\omega, o'}$ with coefficient $\lambda = 1/2$ both preserve γ , $\varphi(\gamma) = \gamma = \varphi'(\gamma)$. Then for $x = \varphi(o') \in \gamma$ we have $|ox| = |oo'|/2$ and similarly for $x' = \varphi'(o) \in \gamma$ we have $|x'o'| = |oo'|/2$. Furthermore, the length of the segment of γ between o, x is half of the length of the segment between o, o' , $L([ox]_\gamma) = L([oo']_\gamma)/2$. Thus $L([xo']_\gamma) = L([oo']_\gamma)/2$ by additivity of the length. Then $L([x'o']_\gamma) = L([oo']_\gamma)/2 = L([xo']_\gamma)$ and hence $x' = x$ by monotonicity of the length, and x is the required midpoint.

It follows that the segment of γ between its any two points is geodesic. Since γ is invariant under the nontrivial homothety $\varphi \in \Gamma_{\omega, o}$, we see that γ is a Ptolemy line unless it degenerates to a point.

If γ is degenerate, then any Busemann function $b : X_\omega \rightarrow \mathbb{R}$ is constant along γ . By Lemma 5.5, we have $\sum_i \alpha_i s_i = 0$ for every oriented Ptolemy line $l \subset X_\omega$, where $\alpha_i = \text{slope}(l_i, l)$, $i = 1, \dots, k$. Conversely, if γ is non-degenerate, then $l = \gamma(\mathbb{R})$ is a Ptolemy line in X_ω by the first part of the proof, and the associated Busemann function $b : X_\omega \rightarrow \mathbb{R}$ is nonconstant along l . By Lemma 5.5, we have $b \circ \gamma(t) = \beta t$ for the canonical parameterization of γ with $\beta = \sum_i \alpha_i s_i / \sum_i s_i$, $\alpha_i = \text{slope}(l_i, l)$, $i = 1, \dots, k$. Thus $\sum_i \alpha_i s_i \neq 0$. \square

Remark 5.9. Assume γ_1 is a piecewise geodesic curve (with finite number of edges) between different fibers in X_ω , $\gamma_1(0) \in F$, $\gamma_1(s_1 + \dots + s_k) \in F'$, $F \neq F'$, where s_1, \dots, s_k are the lengths of its edges. Then the respective zigzag curve $\gamma = \lim \gamma_p$ is not degenerate. This follows from $\gamma(0) \in F$, $\gamma(s_1 + \dots + s_k) \in F'$ by construction of the approximating sequence $\gamma_1, \dots, \gamma_p, \dots \rightarrow \gamma$.

Now, we compute a unit speed parameterization a zigzag curve $\gamma = \gamma(o, \mathcal{L}, \mathcal{S})$ assuming for simplicity that the collection \mathcal{L} consists of mutually orthogonal Ptolemy lines.

Lemma 5.10. *Let $\mathcal{L} = \{l_1, \dots, l_k\}$ be a collection of mutually orthogonal oriented Ptolemy lines in X_ω , $l_i \perp l_j$ for $i \neq j$, $\mathcal{S} = \{s_1, \dots, s_k\}$ a collection*

of nonnegative numbers with $\sum_i s_i = 1$. Then for every $o \in X_\omega$ the zigzag curve $\gamma = \gamma(o, \mathcal{L}, S)$ is nondegenerate, for the Ptolemy line $l = \gamma(\mathbb{R})$ we have

$$\text{slope}(l, l_i) = -s_i/\lambda$$

for every $i = 1, \dots, k$, where $\lambda = \sqrt{\sum_i s_i^2}$, and $|o\gamma(t)| = \lambda|t|$, $t \in \mathbb{R}$, for the canonical parameterization $t \mapsto \gamma(t)$ of γ .

Proof. By Proposition 5.8, $l = \gamma(\mathbb{R})$ is a Ptolemy line or it degenerates to a point. We put $\lambda = |o\gamma(1)| \geq 0$. Let b_i , $i = 1, \dots, k$, be the Busemann function of l_i normalized by $b_i(o) = 0$. By Lemma 5.5,

$$b_i \circ \gamma(1) = \sum_j \text{slope}(l_j, l_i) s_j = -s_i$$

because the lines of \mathcal{L} are mutually orthogonal. Since $\sum_i s_i = 1$ there is i with $s_i \neq 0$. Hence γ is nondegenerate and $\lambda > 0$.

On the other hand, by definition of $\alpha_i = \text{slope}(l, l_i)$ we have $b_i \circ \gamma(1) = \alpha_i |o\gamma(1)| = \alpha_i \lambda$. Thus $\alpha_i \lambda = -s_i$ for every $i = 1, \dots, k$.

Let b be the Busemann function of l normalized by $b(o) = 0$ and $b \circ \gamma(t) < 0$ for $t > 0$. Then using again Lemma 5.5 and the symmetry of the slope, $\alpha_i = \text{slope}(l_i, l)$, we obtain

$$\lambda^2 = \lambda |o\gamma(1)| = -\lambda b \circ \gamma(1) = -\lambda \sum_i \alpha_i s_i = \sum_i s_i^2.$$

Thus $\text{slope}(l, l_i) = -s_i/\lambda$ with $\lambda = \sqrt{\sum_i s_i^2}$.

The shift $\eta = \eta_{o\gamma(1)} : X_\omega \rightarrow X_\omega$ leaves l invariant and $\eta \circ \gamma(1) = \gamma(2)$ by construction of $\gamma = \gamma(o, \mathcal{L}, S)$. Thus $|o\gamma(2)| = 2|o\gamma(1)| = 2\lambda$. By Lemma 5.6, $\varphi \circ \gamma(1) = \gamma(1/2)$, where $\varphi \in \Gamma_{\omega, o}$ is the homothety with coefficient $1/2$. Thus $|o\gamma(1/2)| = \lambda/2$. From this one easily sees that

$$|o\gamma(t)| = \lambda|t|$$

first for dyadic and then by continuity for all $t \in \mathbb{R}$. □

5.3 Orthogonalization procedure

As usual, we fix $\omega \in X$ and a metric d of the Möbius structure with infinitely remote point ω .

Proposition 5.11. *There is a finite collection \mathcal{L}_\perp of mutually orthogonal Ptolemy lines such that for every $x \in X_\omega$ the fiber $F \subset X_\omega$ through x of the fibration $\pi_\omega : X_\omega \rightarrow B_\omega$ is represented as $F = \bigcap_{l \in \mathcal{L}_\perp} H_l$, where H_l is the horosphere of l through x .*

We first note that the cardinality of any collection of mutually orthogonal Ptolemy lines in X_ω is uniformly bounded above.

Lemma 5.12. *There is $N \in \mathbb{N}$ such that the cardinality of any collection \mathcal{L} of mutually orthogonal Ptolemy lines in X_ω is bounded by N , $|\mathcal{L}| \leq N$.*

Proof. We fix $x \in X_\omega$ and assume W.L.G. that all the lines of \mathcal{L} pass through x . On every line $l \in \mathcal{L}$, we fix a point at the distance 1 from x . Let $A \subset X_\omega$ be the set of obtained points. By compactness of X and homogeneity of X_ω it suffices to show that the distance $|aa'| \geq 1$ for each distinct $a, a' \in A$. We have $a \in l, a' \in l'$ for some distinct lines $l, l' \in \mathcal{L}$. Since $\text{slope}(l', l) = 0$, the lines l, l' are also orthogonal at the infinite remote point ω according Lemma 4.4 and Lemma 4.5. Thus in the space X_x with infinitely remote point x , the Ptolemy line $(l' \cup \omega) \setminus x \subset X_x$ lies in the horosphere H of the line $(l \cup \omega) \setminus x \subset X_x$ through ω . By duality, see Lemma 2.12, the point $x \in l$ is closest on the line l to any fixed point of $l' \subset X_\omega$, in particular $|a'a| \geq |a'x| = 1$. \square

Next, we describe an orthogonalization procedure.

Lemma 5.13. *Let l_1, \dots, l_k be a collection of mutually orthogonal Ptolemy lines in X_ω , $l_i \perp l_j$ for $i \neq j$. Given a Ptolemy line $l \subset X_\omega$, through any $o \in X_\omega$ there is a zigzag curve $\gamma = \gamma(o, \mathcal{L}, S)$, where $\mathcal{L} = \{l_1, \dots, l_k, l\}$ is an ordered collection of oriented Ptolemy lines, $S = \{s_1, \dots, s_{k+1}\}$ a collection of nonnegative numbers with $s_1 + \dots + s_{k+1} > 0$, which is orthogonal to l_1, \dots, l_k , $\gamma(\mathbb{R}) = l_{k+1} \perp l_i$ for $i = 1, \dots, k$. Furthermore, if $\sum_1^k \alpha_i^2 \neq 1$, where $\alpha_i = \text{slope}(l, l_i)$, then γ is nondegenerate, and l_{k+1} is a Ptolemy line.*

Proof. We fix an orientation of l and for every $i = 1, \dots, k$ we choose an orientation of l_i so that $\alpha_i = \text{slope}(l, l_i) \geq 0$, and put $\alpha := \sum_i \alpha_i \geq 0$. For any zigzag curve $\gamma = \gamma(o, \mathcal{L}, S)$ in X_ω , where $\mathcal{L} = \{l_1, \dots, l_k, l\}$, $S = \{s_1, \dots, s_{k+1}\}$, for $i = 1, \dots, k$ and for the Busemann function b_i of l_i with $b_i(o) = 0$, by Lemma 5.5 we have $b_i \circ \gamma(t) = \beta_i t$ for all $t \in \mathbb{R}$, where $\beta_i = (-s_i + \alpha_i s_{k+1}) / (s_1 + \dots + s_{k+1})$. Then putting $s_i = \frac{\alpha_i}{1+\alpha}$, $i = 1, \dots, k$, $s_{k+1} = \frac{1}{1+\alpha}$, we have $s_1 + \dots + s_{k+1} = 1$ and $\beta_i = 0$ for every $i = 1, \dots, k$. Thus γ is orthogonal to l_1, \dots, l_k , and it gives us a required Ptolemy line $l_{k+1} = \gamma(\mathbb{R})$ unless γ degenerates.

Let b be the Busemann function of l with $b(o) = 0$. By Lemma 5.5 we have $b \circ \gamma(t) = \beta t$ for all $t \in \mathbb{R}$ with $\beta = \sum_1^k \text{slope}(l_i, l) s_i - s_{k+1}$. Using the symmetry of the slope, we see that $\text{slope}(l_i, l) = \text{slope}(l, l_i) = \alpha_i$ and thus $\beta = (\sum_1^k \alpha_i^2 - 1) / (1 + \alpha)$. If $\sum_1^k \alpha_i^2 \neq 1$, then this shows that γ is nondegenerate. \square

We say that a collection $\{l_1, \dots, l_k\}$ mutually orthogonal Ptolemy lines in X_ω is *maximal* if there is no Ptolemy line in X_ω which is orthogonal to every l_1, \dots, l_k . By Lemma 5.12, such a collection exists.

Lemma 5.14. *Let $\{l_1, \dots, l_k\}$ be a maximal collection of mutually orthogonal Ptolemy lines in X_ω . Then every Ptolemy line $l \subset X_\omega$ can be represented as a zigzag curve $\gamma(o, \mathcal{L}, S)$ with $o \in l$ for a collection $\mathcal{L} = \{l_1, \dots, l_k\}$ of oriented Ptolemy lines and a collection $S = \{s_1, \dots, s_k\}$ of nonnegative numbers with $s_1 + \dots + s_k > 0$. Furthermore, we have $\sum_i \alpha_i^2 = 1$, where $\alpha_i = \text{slope}(l, l_i)$, $i = 1, \dots, k$.*

Proof. We apply the orthogonalization procedure described in Lemma 5.13 to the collection $\mathcal{L}_l = \{l_1, \dots, l_k, l\}$ and construct a zigzag curve $\gamma_l = \gamma_l(o, \mathcal{L}_l, S_l)$, where $S_l = \{s_1, \dots, s_{k+1}\}$ is the collection of nonnegative numbers described there. Since γ_l is orthogonal to l_1, \dots, l_k , we conclude from maximality of $\{l_1, \dots, l_k\}$ that γ_l degenerates and moreover $\sum_i \alpha_i^2 = 1$ by Lemma 5.13.

According to Remark 5.9, the ends of the piecewise geodesic curve $\gamma_{l,1}$ with $k+1$ edges $\sigma_1, \dots, \sigma_k, \sigma$ on Ptolemy lines Busemann parallel to l_1, \dots, l_k, l respectively with $|\sigma_i| = s_i$, $i = 1, \dots, k$, $|\sigma| = s_{k+1}$, lie in one and the same fiber (\mathbb{K} -line) $F \subset X_\omega$, that is, $o = \gamma_{l,1}(0)$ and $x = \gamma_{l,1}(s + s_{k+1}) \in F$, where $s = s_1 + \dots + s_k$. Thus the reduced piecewise geodesic curve $\gamma_1 = \sigma_1 \cup \dots \cup \sigma_k$ and the last edge σ of $\gamma_{l,1}$ have the ends in the same fibers F, F' , where F' is the fiber through $\gamma_{l,1}(s) = \gamma_1(s)$.

Then the zigzag curve $\gamma = \gamma(o, \mathcal{L}, S)$ is nondegenerate, where $\mathcal{L} = \{l_1, \dots, l_k\}$, $S = \{s_1, \dots, s_k\}$, and it gives a Ptolemy line $\gamma(\mathbb{R}) \subset X_\omega$ though o which hits the fiber F' . Since the Ptolemy line l' containing the segment σ is Busemann parallel to l and intersects the fibers F, F' , the line $\gamma(\mathbb{R})$ is Busemann parallel to l , see Lemma 5.3. Hence $\gamma(\mathbb{R}) = l$ by uniqueness, see Lemma 3.9. \square

Lemma 5.15. *Let $\mathcal{L} = \{l_1, \dots, l_k\}$ be a maximal collection of mutually orthogonal oriented Ptolemy lines in X_ω , $S = \{s_1, \dots, s_k\}$ a collection of nonnegative numbers with $s_1 + \dots + s_k = 1$, $b, b_i : X_\omega \rightarrow \mathbb{R}$ the Busemann functions of the zigzag curve $\gamma = \gamma(o, \mathcal{L}, S)$, Ptolemy line l_i with $b(o) = 0 = b_i(o)$ respectively, $i = 1, \dots, k$. Then $\lambda b = \sum_i s_i b_i$, where $\lambda = \sqrt{\sum_i s_i^2}$.*

Proof. We denote by $h = \lambda b - \sum_i s_i b_i$ the function $X_\omega \rightarrow \mathbb{R}$ with $h(o) = 0$, which is an affine function on every Ptolemy line on X_ω . First, we check that h vanishes along l_1, \dots, l_k (assuming that these lines pass through o). Indeed, $\sum_i s_i b_i(z) = s_j b_j(z)$ for every $z \in l_j$ because $l_i \perp l_j$ for $i \neq j$. Since b is a Busemann function of l , it is affine on l_j with the coefficient $\text{slope}(l_j, l)$, $b(z) = -\text{slope}(l_j, l) b_j(z)$. Using symmetry of the slope, we obtain

$$\text{slope}(l_j, l) = \text{slope}(l, l_j) = -s_j/\lambda,$$

see Lemma 5.10. Thus $\lambda b(z) = s_j b_j(z)$, and $h(z) = 0$.

Next, we show that if h is constant on a Ptolemy line l , then it is constant on every Ptolemy line l' that is Busemann parallel to l (with maybe

a different value). By Lemma 3.10 we know that l, l' diverge at most sub-linearly, and also that h is affine on l' . Thus $h|l'$ cannot be nonconstant because h is a Lipschitz function on X_ω .

It follows that h vanishes on every piecewise geodesic curve with origin o and with edges Busemann parallel to the lines l_1, \dots, l_k . Hence, h vanishes along any zigzag curve of type $\gamma(o, \mathcal{L}, S)$. Using Lemma 5.14, we conclude that h is constant along any Ptolemy line in X_ω .

By Proposition 4.6, every Ptolemy circle possesses a unique tangent line, which is certainly a Ptolemy line, at every point. Using standard approximation arguments, we see that h is constant along any Ptolemy circle in X_ω . By the existence property (E), every $x \in X_\omega$ is connected with o by a Ptolemy circle. Thus $h(x) = 0$ and $\lambda b = \sum_i b_i$. \square

Proof of Proposition 5.11. Let $\mathcal{L}_\perp = \{l_1, \dots, l_k\}$ be a maximal collection of mutually orthogonal Ptolemy lines in X_ω . This means that we actually consider respective foliations of X_ω by Busemann parallel Ptolemy lines. For any $x \in X_\omega$, for the fiber F and for the respective lines from \mathcal{L}_\perp through x , we have by definition $F \subset \cap_j H_j$, where $H_j \subset X_\omega$ is the horosphere of l_j through x . It follows from Lemma 5.14 and Lemma 5.15 that any Busemann function $b : X_\omega \rightarrow \mathbb{R}$ with $b(x) = 0$ is a linear combination of the Busemann functions b_1, \dots, b_k of the lines l_1, \dots, l_k which vanish at x . Thus b vanishes on $\cap_j H_j$, and therefore $F = \cap_j H_j$. \square

Proof the property (K). Given a fiber (\mathbb{K} -line) $F \subset X_\omega$ and $x \in X_\omega \setminus F$, we show that there is a Ptolemy line $l \subset X_\omega$ through x that hits F . Uniqueness of l is proved above just after Lemma 5.2.

Using Proposition 5.11, we represent $F = \cap_{l \in \mathcal{L}_\perp} H_l$, where \mathcal{L}_\perp is a finite collection of mutually orthogonal Ptolemy lines, H_l a horosphere of l . Choosing appropriate orientations of the members of \mathcal{L}_\perp , we can assume that $H_l = b_l^{-1}(0)$ and $b_l(x) \geq 0$ for every $l \in \mathcal{L}_\perp$, where $b_l : X_\omega \rightarrow \mathbb{R}$ is a Busemann function of l . Moving from x in an appropriate direction along a Ptolemy line, which is Busemann parallel to $l \in \mathcal{L}_\perp$ with $b_l(x) > 0$, we reduce the value of b_l to zero keeping up every other Busemann function $b_{l'}, l' \in \mathcal{L}_\perp$, constant. Repeating this procedure, we connect x with F by a piecewise geodesic curve with at most $|\mathcal{L}_\perp|$ edges. Now, the zigzag construction produces a required Ptolemy line through x that hits F . \square

5.4 Properties of the base B_ω

We fix $\omega \in X$ and a metric d from the Möbius structure for which ω is infinitely remote. We also use notation $|xy| = d(x, y)$ for the distance between $x, y \in X_\omega$.

Lemma 5.16. *Let $\varphi : X_\omega \rightarrow X_\omega$ be a pure homothety with $\varphi(o) = o$, $o \in X_\omega$. Then φ preserves the fiber (\mathbb{K} -line) F through o , $\varphi(F) = F$. In particular, every shift $\eta_{xx'} : X_\omega \rightarrow X_\omega$ with $x, x' \in F$ preserves F .*

Proof. As in Lemma 3.13, we have $\lambda b \circ \varphi = b$ for any Busemann function $b : X_\omega \rightarrow \mathbb{R}$ with $b(o) = 0$, where λ is the coefficient of the homothety φ . Since $b(x) = 0$ for every $x \in F$, we see that $b \circ \varphi(x) = 0$, and thus $\varphi(x) \in F$, that is, $\varphi(F) = F$. The assertion about a shift $\eta_{xx'}$ follows now from the definition of $\eta_{xx'}$. \square

Lemma 5.17. *Let $\eta : X_\omega \rightarrow X_\omega$ be a shift that preserves a \mathbb{K} -line $F \subset X_\omega$. Then η preserves any other \mathbb{K} -line $F' \subset X_\omega$.*

Proof. Let $b : X_\omega \rightarrow \mathbb{R}$ be a Busemann function associated with an (oriented) Ptolemy line $l \subset X_\omega$. Then for any isometry $\eta : X_\omega \rightarrow X_\omega$ the function $b \circ \eta$ is a Busemann function associated with Ptolemy line $\eta^{-1}(l)$. Thus for an arbitrary shift $\eta : X_\omega \rightarrow X_\omega$, we have $b \circ \eta = b + c_b$, where $c_b \in \mathbb{R}$ is a constant depending on b , because the line $\eta^{-1}(l)$ is Busemann parallel to l , hence the function $b \circ \eta$ is also a Busemann function of l , and thus the functions $b, b \circ \eta$ differ by a constant.

In our case, when η preserves a \mathbb{K} -line, this constant is zero, $c_b = 0$, thus $b \circ \eta = b$ for any Busemann function $b : X_\omega \rightarrow \mathbb{R}$. Therefore, η preserves any \mathbb{K} -line. \square

We define

$$d(F, F') = \inf\{|xx'| : x \in F, x' \in F'\}$$

for \mathbb{K} -lines $F, F' \subset X_\omega$.

Lemma 5.18. *Given \mathbb{K} -lines $F, F' \subset X_\omega$, and $x \in F$, there is $x' \in F'$ such that $d(F, F') = |xx'|$.*

Proof. Let $x_i \in F, x'_i \in F'$ be sequences with $|x_i x'_i| \rightarrow d(F, F')$. Using Lemma 5.17, we can assume that $x_i = x$ for all i . Then the sequence x'_i is bounded, and by compactness of X it subconverges to $x' \in F'$ with $|xx'| = d(F, F')$. \square

Lemma 5.19. *For any \mathbb{K} -lines $F, F' \subset X_\omega$, we have $d(F, F') = |xx'|$, where $x = l \cap F, x' = l \cap F'$, and l is any Ptolemy line in X_ω that meets both F, F' .*

Proof. By Lemma 5.3, the distance $|xx'|$ is independent of the choice of l . By definition $|xx'| \geq d(F, F')$. By Lemma 5.18, there is $x'' \in F'$ with $|xx''| = d(F, F')$. The horosphere H of (a Busemann function associated with) l through x' contains F' , in particular, $x'' \in H$. Then $|xx''| \geq |xx'|$ and hence, $d(F, F') = |xx'|$. \square

Let $\pi_\omega : X_\omega \rightarrow B_\omega$ be the canonical fibration, see sect. 5. For $b \in B_\omega$, we denote by $F_b = \pi_\omega^{-1}(b)$ the \mathbb{K} -line over b . For $b, b' \in B_\omega$ we put $|bb'| := |xx'|$, where $x = l \cap F_b$, $x' = l \cap F_{b'}$, and $l \subset X_\omega$ is any Ptolemy line that meets both F_b and $F_{b'}$. By property (\mathbb{K}) , such a line l exists, by Lemma 5.19, the number $|bb'|$ is well defined, and the function $(b, b') \mapsto |bb'|$ is a metric on B_ω . This metric is said to be *canonical*.

Proposition 5.20. *The canonical projection $\pi_\omega : X_\omega \rightarrow B_\omega$ is a 1-Lipschitz submetry with respect to the canonical metric on B_ω . Furthermore, B_ω is a geodesic metric space with the property that through any two distinct points $b, b' \in B_\omega$ there is a unique geodesic line in B_ω .*

Proof. It follows from Lemma 5.19 that the map π_ω is 1-Lipschitz. Let $D = D_r(o)$ be the metric ball in X_ω of radius $r > 0$ centered at a point $o \in X_\omega$, $D' \subset B_\omega$ the metric ball of the same radius r centered at $\pi_\omega(o)$. The inclusion $D' \subset \pi_\omega(D)$ follows from the definition of the metric of B_ω . The opposite inclusion $D' \supset \pi_\omega(D)$ holds because π_ω is 1-Lipschitz. Thus $\pi_\omega : X_\omega \rightarrow B_\omega$ is a 1-Lipschitz submetry.

Furthermore, by Lemma 5.19, the projection π_ω restricted to every Ptolemy line in X_ω is isometric, and thus by property (\mathbb{K}) , the base B_ω is a geodesic metric space. Moreover, it follows from (\mathbb{K}) that through any two distinct points $b, b' \in B_\omega$ there is a unique geodesic line in B_ω . \square

Corollary 5.21. *For any homothety $\varphi : X_\omega \rightarrow X_\omega$, the induced map $\pi_*(\varphi) : B_\omega \rightarrow B_\omega$ is a homothety with the same dilatation coefficient.* \square

Proposition 5.22. *The base B_ω is isometric to an Euclidean \mathbb{R}^k for some $k > 0$.*

Proof. Any Busemann function $b : X_\omega \rightarrow \mathbb{R}$ is affine on Ptolemy lines by Corollary 3.19. By definition, b is constant on the fibers of π_ω , thus it determines a function $\bar{b} : B_\omega \rightarrow \mathbb{R}$ such that $\bar{b} \circ \pi_\omega = b$. This function is affine on geodesic lines in B_ω because every geodesic line $\bar{l} \subset B_\omega$ is of the form $\bar{l} = \pi_\omega(l)$ for some Ptolemy line $l \subset X_\omega$, and each unit speed parameterization $c : \mathbb{R} \rightarrow X_\omega$ of l induces the unit speed parameterization $\bar{c} = \pi_\omega \circ c$ of \bar{l} . Then $\bar{b} \circ \bar{c} = \bar{b} \circ \pi_\omega \circ c = b \circ c$ is an affine function on \mathbb{R} .

We fix a base point $o \in X_\omega$ and a maximal collection $\mathcal{L} = \{l_1, \dots, l_k\}$ of mutually orthogonal oriented Ptolemy lines of X_ω through o . Let b_1, \dots, b_k be Busemann functions of the lines l_1, \dots, l_k respectively that vanish at o . We denote by \bar{l}_i the projection of l_i to B_ω , and by $\bar{b}_i : B_\omega \rightarrow \mathbb{R}$ the function corresponding to b_i , $i = 1, \dots, k$. By Proposition 5.11, the functions b_1, \dots, b_k separates fibers in X_ω . Thus the functions $\bar{b}_1, \dots, \bar{b}_k$ separates points of B_ω , that is, for each $z, z' \in B_\omega$ there is i with $\bar{b}_i(z) \neq \bar{b}_i(z')$. Therefore, the continuous map $h : B_\omega \rightarrow \mathbb{R}^k$, $h(z) = (\bar{b}_1(z), \dots, \bar{b}_k(z))$ is injective. This map is surjective by the same argument as in the proof of the property (\mathbb{K}) , and it introduces coordinates on B_ω . We compute

the distance on B_ω in these coordinates. Applying a shift if necessary, see Corollary 5.21, we consider W.L.G. the distance $|\bar{o}\bar{z}|$ for every $z \in X_\omega$, where $\bar{z} = \pi_\omega(z)$. By (K) there is a unique Ptolemy line $l \subset X_\omega$ through o that hits the fiber F_z through z . It follows from our definitions that for $\alpha_i = \text{slope}(l, l_i)$ we have $\bar{b}_i(\bar{z}) = \alpha_i |\bar{o}\bar{z}|$, $i = 1, \dots, k$. By Lemma 5.14, $\sum_i \alpha_i^2 = 1$, thus $|\bar{o}\bar{z}|^2 = \sum_i \bar{b}_i^2(\bar{z})$. This shows that B_ω is isometric to an Euclidean \mathbb{R}^k . We have $k > 0$, because there is a Ptolemy line in X_ω . \square

This completes the proof of Theorem 3.4.

6 Lie group structure of X_ω and of \mathbb{K} -lines

Here we recover a natural group structure on every space X_ω , $\omega \in X$, which is a simply connected nilpotent group Lie.

6.1 Groups of shifts

Recall that by Lemma 3.13 a shift $\eta : X_\omega \rightarrow X_\omega$ is an isometry that preserves every foliation of X_ω by (oriented) Busemann parallel Ptolemy lines. Clearly, the shifts of X_ω form a group which we denote by N_ω . Then N_ω is a subgroup of the group $\text{Aut } X$ of the Möbius automorphisms of X .

Lemma 6.1. *Let $V \subset X$ be a closed subset containing with every point $z \in V$, $z \neq \omega'$, every Ptolemy circle through z and a fixed point $\omega' \in X$ (we assume that V contains at least two points). Then $V = X$.*

Proof. Assume there is $\omega \in X \setminus V$. Since V is closed, ω is contained in $X \setminus V$ together with some its neighborhood. Then V is compact in the space X_ω . Thus the image $\bar{V} = \pi_\omega(V) \subset B_\omega$ under the canonical projection $\pi_\omega : X_\omega \rightarrow B_\omega$ is compact. By Proposition 5.22, B_ω is isometric to \mathbb{R}^k for $k \geq 1$, thus we can find a hyperplane $E \subset B_\omega$ supporting to \bar{V} at some point $\bar{z} \in \bar{V}$, i.e. $\bar{z} \in E$ and \bar{V} is contained in a closed half-space of B_ω bounded by E . Let $\bar{l} \subset B_\omega$ be a geodesic line through \bar{z} that is transversal to E . We take $z \in V$ with $\pi_\omega(z) = \bar{z}$, and let $l \subset X_\omega$ be the Ptolemy line through z with $\pi_\omega(l) = \bar{l}$. By Corollary 4.7 there is a (unique) Ptolemy circle $\sigma \subset X_\omega$ through z and ω' that is tangent to l (note that l misses ω' since otherwise $l \subset V$ in contradiction with compactness of V .) By the assumption, $\sigma \subset V$ and thus $\bar{\sigma} = \pi_\omega(\sigma) \subset \bar{V}$. This is a contradiction, because the curve $\bar{\sigma}$ is tangent to \bar{l} at \bar{z} , and therefore $\bar{\sigma} \not\subset \bar{V}$. \square

Proposition 6.2. *The group N_ω acts simply transitively on X_ω .*

Proof. Given $x, x' \in X_\omega$, the shift $\eta_{xx'}$ moves x to x' , $\eta_{xx'}(x) = x'$, by construction, see sect. 3.3. Thus N_ω acts transitively on X_ω .

Assume that $\eta(x) = x$ for some shift $\eta : X_\omega \rightarrow X_\omega$ and some $x \in X_\omega$. We denote by V the fixed point set of η , $\eta(y) = y$ for every $y \in V$ (in particular,

$\omega \in V$). We show that $V = X$. Note that every Ptolemy line $l \subset X_\omega$, which meets V , is contained in V because the isometry η preserves every foliation of X_ω by Busemann parallel Ptolemy lines. Applying Lemma 6.1, we obtain $V = X$, and thus $\eta = \text{id}$, i.e. the group N_ω acts simply transitively on X_ω . \square

We fix $o \in X_\omega$ and using Proposition 6.2 identify N_ω with X_ω by $\eta \mapsto \eta(o)$. Then N_ω is a locally compact topological group.

An automorphism $\tau : N_\omega \rightarrow N_\omega$ is said to be *contractible* if for every $\eta \in N_\omega$ we have $\lim_{n \rightarrow \infty} \tau^n(\eta) = \text{id}$. If N_ω admits a contractible automorphism, then N_ω is also said to be *contractible*.

Lemma 6.3. *There is a contractible automorphism $\tau : N_\omega \rightarrow N_\omega$.*

Proof. We take any pure homothety $\varphi : X_\omega \rightarrow X_\omega$ with $\varphi(o) = o$ and with the coefficient $\lambda \in (0, 1)$. Then we define $\tau(\eta) = \varphi \circ \eta \circ \varphi^{-1}$. The map $\eta' = \tau(\eta) : X_\omega \rightarrow X_\omega$ is an isometry preserving every foliation of X_ω by Busemann parallel Ptolemy lines, i.e. η' is a shift, and it is clear that τ is an automorphism of N_ω .

For the sequence of shifts $\eta_n = \tau^n(\eta)$ we have $\eta_n(o) = \varphi^n \circ \eta(o) \rightarrow o$ as $n \rightarrow \infty$. Thus η_n converges to a shift η_∞ with $\eta_\infty(o) = o$, hence, $\eta_\infty = \text{id}$. \square

Corollary 6.4. *The group N_ω is a simply connected nilpotent Lie group. In particular, the space X_ω is homeomorphic to \mathbb{R}^n , and the space X is homeomorphic to the sphere S^n with $n = \dim X$. Furthermore, every metric ball $B = B_r(o) = \{x \in X_\omega : |xo| \leq r\}$ in X_ω is homeomorphic to the standard ball in \mathbb{R}^n .*

Proof. The group N_ω is connected and locally compact because the space X_ω is. By Lemma 6.3, N_ω is contractible. Then by [Sieb, Corollary 2.4] N_ω is a simply connected nilpotent Lie group. Thus $X_\omega \simeq \mathbb{R}^n$.

Let $h_\lambda : X_\omega \rightarrow X_\omega$ be the pure homothety with coefficient $\lambda > 0$ centered at $o \in X_\omega$, $h_\lambda(o) = o$, $h_1 = \text{id}$. Then $B = o \cup \{h_\lambda(S) : 0 < \lambda \leq 1\}$, where $S = \partial B = \{x \in X_\omega : |xo| = r\}$. This gives a representation $\text{int } B = o \cup \{(x, \lambda) : x \in S, 0 < \lambda < 1\}$. Composing the homeomorphism $g : [0, 1) \rightarrow [0, \infty)$, $g(\lambda) = \tan \frac{\lambda\pi}{2}$ with the embedding $f_x : [0, \infty) \rightarrow X_\omega$, $f_x(0) = o$, $f_x(\lambda) = h_\lambda(x)$ for every $x \in S$ we obtain a homeomorphism $F : \text{int } B \rightarrow X_\omega$, $F(x, \lambda) = f_x \circ g(\lambda)$. Hence $\text{int } B$ is homeomorphic to $X_\omega \simeq \mathbb{R}^n$ and B is homeomorphic to the standard ball in \mathbb{R}^n . \square

We denote by Z_ω a subgroup in N_ω which consists of all shifts $\eta \in N_\omega$ acting identically on the base B_ω , $\pi_*(\eta) = \text{id}$, where $\pi_*(\eta) : B_\omega \rightarrow B_\omega$ is the shift induced by the projection $\pi_\omega : X_\omega \rightarrow B_\omega$, see Corollary 5.21. Every $\eta \in Z_\omega$ preserves every fiber (\mathbb{K} -line) of π_ω , see Lemma 5.16 and Lemma 5.17.

Proposition 6.5. *The group Z_ω acts simply transitively on every \mathbb{K} -line $F \subset X_\omega$, and thus it is a contractible, connected, locally compact topological group. Therefore, Z_ω is a simply connected nilpotent Lie group, and F is homeomorphic to \mathbb{R}^p for some $0 \leq p < n$, $k + p = n$ for $k = \dim B_\omega$.*

Proof. The group Z_ω acts transitively on F by Lemma 5.16. The action is simply transitive by Proposition 6.2. We fix $o \in F$ and identify Z_ω with F by $\eta \mapsto \eta(o)$. By the same argument as in Lemma 6.3 we see that the group Z_ω is contractible. Furthermore, F is locally compact. Given $x, x' \in F$, there is a Ptolemy circle $\sigma \subset X_\omega$ through x, x' . By (\mathbb{K}) , through any point $z \in \sigma$ there is a uniquely determined Ptolemy line that hits F . This defines a continuous map $\sigma \rightarrow F$. Thus F is linearly connected. Hence, Z_ω is a contractible, locally compact, connected topological group. By [Sieb, Corollary 2.4], Z_ω is a simply connected nilpotent Lie group, and thus F is homeomorphic to \mathbb{R}^p for some $0 \leq p \leq n$. In fact $p < n$ because X contains Ptolemy circles and thus $k = \dim B_\omega > 0$, while $n = k + p$. \square

6.2 Non-integrability of the canonical distribution

Given $o \in X_\omega$, $\bar{x} \in B_\omega$, by the property (\mathbb{K}) there is a unique $x \in F_{\bar{x}} = \pi_\omega^{-1}(\bar{x})$ that is connected with o by a geodesic segment ox . The point x is called the *lift* of \bar{x} with respect to o , and we use notation $x = \text{lift}_o(\bar{x})$. This defines an embedding $\text{lift}_o : B_\omega \rightarrow X_\omega$ with $\pi_\omega \circ \text{lift}_o = \text{id}$ for every $o \in X_\omega$. We denote $D_o = \text{lift}_o(B_\omega)$. The embedding lift_o is radially isometric, $|\text{lift}_o(\bar{x})| = |\pi(o)\bar{x}|$ for every $\bar{x} \in B_\omega$. Though there is no reason for lift_o as well as for the projection $\pi_\omega|_{D_o}$ to be isometric, the map lift_o is continuous which follows the uniqueness property of (\mathbb{K}) and compactness of X .

The family of subspaces D_o , $o \in X_\omega$, is called the (canonical) *distribution* on X_ω . We say that the canonical distribution $\mathcal{D} = \{D_o : o \in X_\omega\}$ on X_ω is *integrable* if for any $o \in X_\omega$ and any $o' \in D_o$, the subspaces D_o and $D_{o'}$ of X_ω coincide, $D_o = D_{o'}$. For example, if the base B_ω is one-dimensional, then the canonical distribution \mathcal{D} is obviously integrable.

Proposition 6.6. *The canonical distribution on X_ω is integrable for some $\omega \in X$ if and only if $p = 0$, i.e. every fiber of the projection π_ω is a point for every $\omega \in X$. In this case the space X is Möbius equivalent to $\widehat{\mathbb{R}}^n$ with $n = \dim X$.*

Proof. Assume that the canonical distribution is integrable for some $\omega \in X$. Since $D_z = D_o$ for every $o \in X_\omega$ and every $z \in D_o$, every Ptolemy line in X_ω through z is contained in D_o . By Lemma 6.1, $D_o \cup \omega = X$. Hence $p = 0$, $n = k$, and X is Möbius equivalent to $\widehat{\mathbb{R}}^n$ with $n = \dim X$. Conversely, assume $p = 0$, and $o' \in D_o \subset X_\omega$, $\omega \in X$. Then by property (\mathbb{K}) for every $x \in D_{o'}$ there is a unique Ptolemy line in X_ω through x that hits the fiber $F_o = \{o\}$. Hence $D_{o'} \subset D_o$, and similarly $D_o \subset D_{o'}$, that is, \mathcal{D} is integrable for every $\omega \in X$. \square

Corollary 6.7. *Assume $p > 0$, that is, fibers of the canonical projections π_ω , $\omega \in X$, are nondegenerate. Then the canonical distribution on X_ω is non-integrable for every $\omega \in X$. \square*

The next corollary follows immediately from Proposition 6.6.

Corollary 6.8. *If the base B_ω of X_ω is one-dimensional (this is independent of $\omega \in X$), then $p = 0$ and $X = \widehat{\mathbb{R}}$. \square*

6.3 Classification of 2-transitive actions and Theorem 1.1

By Proposition 3.15, the group $\text{Aut } X$ of Möbius transformations of X is 2-transitive, see Remark 3.16. It is known from the Tits' classification of 2-transitive actions that:

If a topological group G , acting effectively and 2-transitively on a compact topological space X , which is not totally disconnected, is locally compact and σ -compact (that is, G is a countable union of compact subsets), then G is a Lie group and $X = G/G_x$ is a smooth and connected manifold, homeomorphic to either a projective space, or a sphere S^n . In the second case, G is isomorphic to the isometry group of a rank one symmetric space of non-compact type (up to a subgroup of index 2),

see [Kr, Theorems A, B and 3.3(a)]. Corollary 6.4 excluded the first possibility. Thus to complete the proof of Theorem 1.1, it would be sufficient to check that $\text{Aut } X$ is locally compact and σ -compact, and that the Möbius structure of X is uniquely determined by the respective appropriately normalized symmetric space. However, from our point of view, this formal classification argument is not satisfactory. Instead, we give a direct, classification free proof of Theorem 1.1 by showing how the symmetric space structure emerges from the Möbius structure. This is done in the next two sections.

7 Extension of Möbius automorphisms of circles

The main purpose of this section is to prove the the following extension property.

(E₂) Extension: any Möbius map between any Ptolemy circles in X extends to a Möbius automorphism of X .

Proposition 7.1. *Any compact Ptolemy space with properties (E) and (I) possesses the extension property (E₂).*

The proof of this result is based on study of second order properties of Ptolemy circles like Lemma 7.5.

7.1 Distance and arclength parameterizations of a circle

In this section, we establish existence of a distance parameterization in a Ptolemy circle and study its relationship with an arclength parameterization. A distance parameterization is convenient to obtain an important estimate (9) below. On the other hand, in an application of this estimate we compute slops, and that is most convenient to do in an arclength parameterization.

In what follows, we consider a (bounded) Ptolemy circle $\sigma \subset X_\omega$ and points $x, y \in \sigma$ with $a := |xy| > 0$.

Lemma 7.2. *Let σ_+, σ_- be the two components of $\sigma \setminus \{x, y\}$. Then for all $0 < t < a$ there exists exactly one point $x_t^+ \in \sigma_+$ (resp. $x_t^- \in \sigma_-$) with $|xx_t^+| = |xx_t^-| = t$. Therefore $\gamma : (-a, a) \rightarrow \sigma$ with $\gamma(0) = x$, $\gamma(t) = x_t^+$ for $t > 0$, and $\gamma(t) = x_t^-$ for $t < 0$ parameterizes a neighborhood of x in σ .*

Proof. The existence of a point $x_t^+ \in \sigma_+$ with $|xx_t^+| = t$ is clear by continuity. To prove uniqueness consider points $x < p < q < y$ in this order on σ_+ and assume $b := |xp| < a = |xy|$. Let $c := |xq|$, $\lambda_a := |pq|$, $\lambda_b := |qy|$, $\lambda_c := |py|$.

The Ptolemy equality and the triangle inequality give

$$a\lambda_a + b\lambda_b = c\lambda_c \leq c(\lambda_a + \lambda_b).$$

Therefore

$$c \geq \frac{\lambda_a}{\lambda_a + \lambda_b}a + \frac{\lambda_b}{\lambda_a + \lambda_b}b > b$$

where the last equality holds, since $a > b$. In particular $|xp| \neq |xq|$. \square

In what follows, we use the parameterization $\gamma : (-a, a) \rightarrow \sigma$ of a neighborhood of $x \in \sigma$, and call it a *distance* parameterization.

Lemma 7.3. *The function $g(t) := |\gamma(t)y|$ is concave and C^1 -smooth on $(-a, a)$.*

Proof. For $-a \leq t_1 < t_2 \leq a$ the Ptolemy equality for the points $x, \gamma(t_1), \gamma(t_2), y$ implies

$$t_2g(t_1) - t_1g(t_2) = a|\gamma(t_1)\gamma(t_2)|.$$

Thus for $-a \leq t_1 < t_2 < t_3 \leq a$ the triangle inequality $|\gamma(t_1)\gamma(t_3)| \leq |\gamma(t_1)\gamma(t_2)| + |\gamma(t_2)\gamma(t_3)|$ implies

$$t_3g(t_2) - t_2g(t_3) + t_2g(t_1) - t_1g(t_2) \geq t_3g(t_1) - t_1g(t_3)$$

which is equivalent to

$$\frac{g(t_2) - g(t_1)}{t_2 - t_1} \geq \frac{g(t_3) - g(t_2)}{t_3 - t_2}.$$

Therefore, g is concave. It follows, in particular, that g has the left $g'_-(t)$ and the right derivative $g'_+(t)$ at every $t \in (-a, a)$, $g'_-(t) \geq g'_+(t)$ and

these derivatives are nonincreasing, $g'_+(t) \geq g'_-(t')$ for $t < t'$. Furthermore, $g'_-(t) \rightarrow g'_-(t')$ as $t \nearrow t'$, $g'_+(t') \rightarrow g'_+(t)$ as $t' \searrow t$. These are standard well known facts about concave functions, see e.g. [H-UL].

We fix $t_0 \in (-a, a)$ and consider the Ptolemy line $l \subset X_\omega$ tangent to σ at $x_0 = \gamma(t_0)$. We assume that l is oriented and that its orientation is compatible with the orientation of σ given by the distance parameterization γ . Let $c : \mathbb{R} \rightarrow X_\omega$ be the unit speed parameterization of l compatible with the orientation, $c(0) = x_0$. By Corollary 4.7, $y \notin l$. By Proposition 3.18, the function $\tilde{g}(s) = |c(s)y|$, $s \in \mathbb{R}$, is C^1 -smooth. If $t_0 = 0$, then $g'_-(t_0) = \frac{d\tilde{g}}{ds}(0) = g'_+(t_0)$, because l is tangent to σ at $x_0 = x$, and thus g is differentiable at $t_0 = 0$.

Consider now the case $t_0 \neq 0$. Then again by Corollary 4.7, $x \notin l$, thus the function $\tilde{f}(s) = |c(s)x|$, $s \in \mathbb{R}$, is C^1 -smooth. We show that $\frac{d\tilde{f}}{ds}(0) \neq 0$. We suppose W.L.G. that $t_0 > 0$. Then for all $t_1 \in (0, t_0)$ sufficiently close to t_0 we have $\frac{d\tilde{h}}{ds}(0) \neq 0$, where $\tilde{h}(s) = |x_1 c(s)|$, $x_1 = \gamma(t_1)$. We fix such a point t_1 , and using Lemma 7.2 consider the distance parameterization of a neighborhood of $x_0 = \gamma(t_0)$ in σ , $|z(\tau)x_1| = \tau$ for all $z \in \sigma$ sufficiently close to x_0 . Then $t = t(\tau)$ and the function $f(\tau) = |x\gamma \circ t(\tau)|$ is concave by the first part of the proof. Since the functions $\tilde{f}(s) = |xc(s)|$, $\tilde{h}(s) = |x_1 c(s)|$ are C^1 -smooth, and $\frac{d\tilde{h}}{ds}(0) \neq 0$, the function $\tilde{f}(\tau) = \tilde{f} \circ \tilde{h}^{-1}(\tau)$ is C^1 -smooth in a neighborhood of $\tau_0 = |x_1 x_0|$ by the inverse function theorem. Therefore, $f'_-(\tau_0) = \frac{d\tilde{f}}{ds}(0) = f'_+(\tau_0)$ because l is tangent to σ at x_0 . The assumption $\frac{d\tilde{f}}{ds}(0) = 0$ implies $\frac{df}{d\tau}(\tau_0) = 0$. By concavity, τ_0 is a maximum point of the function $f(\tau)$, and there are different τ, τ' arbitrarily close to τ_0 with $f(\tau) = f(\tau')$. This contradicts properties of the parameterization $\gamma(\tau) = \gamma \circ t(\tau)$. Hence, $\frac{d\tilde{f}}{ds}(0) \neq 0$.

Again, by the inverse function theorem, the function $\tilde{g}(t) = \tilde{g} \circ \tilde{f}^{-1}(t)$ is C^1 -smooth in a neighborhood of t_0 . However, $\frac{d\tilde{g}}{dt}(t_0)$ coincides with the left as well as with the right derivative of the function g at t_0 because l is tangent to σ at x_0 . Therefore, g is differentiable at t_0 . It follows from continuity properties of one-sided derivatives of concave functions that the derivative g' is continuous, i.e., g is C^1 -smooth. \square

Lemma 7.4. *Every Ptolemy circle $\sigma \subset X_\omega$ is rectifiable and*

$$L(xx') = |xx'| + o(|xx'|^2)$$

as $x' \rightarrow x$ in σ , where $L(xx')$ is the length of the (smallest) arc $xx' \subset \sigma$.

Proof. We fix $y \in \sigma$, $y \neq x$, and introduce a distance parameterization $\gamma : (-a, a) \rightarrow \sigma$ of a neighborhood of $x = \gamma(0)$ in σ . Rescaling the metric of X_ω we assume that $a = |xy| = 1$ for simplicity of computations. We use notation $d(z, z') = |zz'|$ for the distance in X_ω , and $|zz'|_y$ for the distance

in X_y , assuming that

$$|zz'|_y = \frac{|zz'|}{|zy||z'y|}$$

is the metric inversion of the metric d .

Recall that $\sigma \setminus y$ is a Ptolemy line in X_y . Thus for a given $r \in (0, 1)$, and for every partition $0 = t_0 \leq \dots \leq t_n = r$ we have

$$\sum_i |\gamma(t_i)\gamma(t_{i+1})| \leq \Lambda |xx_r|_y,$$

where $\Lambda = \max\{g^2(t) : 0 \leq t \leq r\}$, $g(t) = |\gamma(t)y|$, $x_r = \gamma(r)$. Hence σ is rectifiable and $L(xx_r) \leq \Lambda |xx_r|_y$. Moreover, using $|\gamma(t_i)\gamma(t_{i+1})| = g(t_i)g(t_{i+1})|\gamma(t_i)\gamma(t_{i+1})|_y$, we actually have

$$L(r) = L(xx_r) = \int_0^{r/g(r)} g^2(s) ds,$$

where $g(s) = g \circ t(s)$ with $s = \frac{t}{g(0)g(t)} = t/g(t)$. Recall that the function $g(t)$ is C^1 -smooth by Lemma 7.3. Then $ds = \frac{g(t)-tg'(t)}{g^2(t)} dt$ and $\frac{dt}{ds} = \frac{g^2(t)}{g(t)-tg'(t)}$, in particular, $\frac{dt}{ds}(0) = 1$.

Using developments $g(s) = 1 + \frac{dg}{ds}(0)s + o(s)$, $g^2(s) = 1 + 2\frac{dg}{ds}(0)s + o(s)$, we obtain

$$L(r) = \frac{r}{g(r)} + \frac{dg}{ds}(0) \frac{r^2}{g^2(r)} + o(r^2),$$

where $\frac{dg}{ds}(0) = g'(0)\frac{dt}{ds}(0) = g'(0)$. Since $g(r) = 1 + g'(0)r + o(r)$, $g^2(r) = 1 + 2g'(0)r + o(r)$, we finally have

$$L(r) = r(1 - g'(0)r) + g'(0)r^2(1 - 2g'(0)r) + o(r^2) = r + o(r^2).$$

Hence $L(xx') = |xx'| + o(|xx'|^2)$ as $x' \rightarrow x$ in σ . □

Lemma 7.5. *Assume that a (bounded) oriented Ptolemy circle $\sigma \subset X_\omega$ has two different points in common with a Ptolemy line $l \subset X_\omega$, $x, y \in \sigma \cap l$, and the line l is oriented from y to x . Let $\sigma_+ \subset \sigma$ be the arc of σ from x to y chosen according to the orientation of σ . Let x_t, y_t be the distance parameterizations of neighborhoods of x, y respectively such that $x_t, y_t \in \sigma_+$ for $t > 0$, $|x_t x| = t = |y_t y|$. Furthermore, let $b^\pm : X_\omega \rightarrow \mathbb{R}$ be the opposite Busemann functions of l normalized by $b^+(x) = 0$, $b^+(y) = -a$, $b^-(x) = -a$, $b^-(y) = 0$, where $a = |xy|$. Then*

$$b^+(x_t) + b^-(y_t) \leq 2\alpha t - \frac{1}{a}(1 - \alpha^2)t^2 \quad (9)$$

for all $t > 0$ in the domain of the parameterizations, where $\alpha = \text{slope}(\sigma, l)$.

Proof. By Lemma 7.3 the functions $g(t) = |x_t y|$, $f(t) = |y_t x|$ are C^1 -smooth and concave. Furthermore, their first derivatives at 0, $g'(0)$ and $f'(0)$, coincide with first derivatives of the distance functions to the respective tangent lines, $g'(0) = \tilde{g}'(0)$ and $f'(0) = \tilde{f}'(0)$, where $\tilde{g}(s) = |c_x(s)y|$, $\tilde{f}(s) = |c_y(s)x|$, and the unit speed parameterizations of the tangent lines l_x to σ at x and l_y to σ at y are chosen compatible with the distance parameterizations x_t , y_t so that $c_x(0) = x$, $c_y(0) = y$.

Using that l is oriented from y to x , and σ_+ from x to y and applying equation (8), we find $\tilde{g}'(0) = \text{slope}(l_x, l)$. By the same equation (8) we have $\tilde{f}'(0) = \text{slope}(l_y, -l) = -\text{slope}(l_y, l)$. The sign -1 appears because the orientation of l from x to y is opposite to the chosen orientation. Note that the orientation of l_x is compatible with that of σ , while the orientation of l_y is opposite to that of σ . Therefore, $g'(0) = \alpha = \text{slope}(\sigma, l)$ and $f'(0) = -\text{slope}(l_y, l) = \text{slope}(\sigma, l) = \alpha$.

Using concavity we obtain $g(t) \leq g(0) + g'(0)t = a + \alpha t$ and similarly $f(t) \leq a + \alpha t$ for all $0 \leq t < a$. The Ptolemy equality applied to the ordered quadruple $(x, x_t, y_t, y) \subset \sigma$ gives $g(t)f(t) = t^2 + a|x_t y_t|$, hence

$$|x_t y_t| \leq a + 2\alpha t - \frac{1}{a}(1 - \alpha^2)t^2.$$

Let H_t^+ , H_t^- be the horospheres of b^+ , b^- through x_t , y_t respectively, $x_t \in H_t^+$, $y_t \in H_t^-$. Since X is Busemann flat, see Proposition 3.18, H_t^+ is also a horosphere of b^- and H_t^- is a horosphere of b^+ . Thus $|x_t y_t| \geq |b^+(x_t) - b^+(y_t)| := \xi$ because Busemann functions are 1-Lipschitz. On the other hand, ξ is the distance between the points $l \cap H_t^+$, $l \cap H_t^-$, and thus $\xi = a + b^+(x_t) + b^-(y_t)$. Therefore,

$$b^+(x_t) + b^-(y_t) \leq 2\alpha t - \frac{1}{a}(1 - \alpha^2)t^2.$$

□

7.2 Proof of the extension property (E₂)

Here we prove Proposition 7.1.

Lemma 7.6. *Any Möbius automorphism of any Ptolemy circle $\sigma \subset X$ preserving orientations extends to a Möbius automorphism of X .*

Proof. We represent σ as the boundary at infinity of the real hyperbolic plane, $\sigma = \partial_\infty \mathbb{H}^2$, so that the Möbius structure of σ induced from X is identified with the canonical Möbius structure of $\partial_\infty \mathbb{H}^2$. Then the group $\text{Aut}_\sigma X$ of preserving orientations Möbius automorphisms of σ is identified with the group of preserving orientations isometries of \mathbb{H}^2 . The last is generated by central symmetries, and any central symmetry of \mathbb{H}^2 induces an s-inversion of σ . Thus $\text{Aut}_\sigma X$ is generated by s-inversions of σ .

Now, any s-inversion of σ can be obtained as follows. Take distinct $\omega, \omega' \in \sigma$ and a metric sphere $S \subset X$ between ω, ω' . Then an s-inversion $\varphi = \varphi_{\omega, \omega', S} : X \rightarrow X$ restricts to an s-inversion of σ . Thus any Möbius automorphism of σ from $\text{Aut}_\sigma X$ extends to a Möbius automorphism of X . \square

The group $\text{Aut } X$ of Möbius automorphisms of X is non-compact: a sequence of homotheties of X_ω with coefficients $\lambda_i \rightarrow \infty$ and with the same fixed point has no converging subsequences. However, we have the following standard compactness result.

Lemma 7.7. *Assume that for a nondegenerate triple $T = (x, y, z) \subset X$ and for a sequence $\varphi_i \in \text{Aut } X$ the sequence $T_i = \varphi_i(T)$ converges to a nondegenerate triple $T' = (x', y', z') \subset X$. Then there exists $\varphi \in \text{Aut } X$ with $\varphi(T) = T'$.*

Proof. For every $u \in X \setminus T$ the quadruple $Q = (T, u)$ is nondegenerate in the sense that its cross-ratio triple $\text{crt}(Q) = (a : b : c)$ has no zero entry. Since $\text{crt}(\varphi_i(Q)) = \text{crt}(Q)$, any accumulation point u' of the sequence $u_i = \varphi_i(u)$ is not in T' . Thus for the nondegenerate triple $S = (x, y, u)$ any sublimit $S' = (x', y', u')$ of the sequence $S_i = \varphi_i(S)$ is nondegenerate. Applying the same argument to any $v \in X \setminus S$, we observe that the sequences $u_i, v_i = \varphi_i(v)$ have no common accumulation point. This shows that any limiting map φ of the sequence φ_i , obtained e.g. by taking a nonprincipal ultra-filter limit, is injective, and hence it is a Möbius automorphism of X with $\varphi(T) = T'$. \square

Proposition 7.8. *Assume $X \neq \widehat{\mathbb{R}}$. Then the group of Möbius automorphisms of X acts transitively on the set of the oriented Ptolemy circles in X . In particular, for any oriented circle $\sigma \subset X$ there is a Möbius automorphism $\varphi : X \rightarrow X$ such that $\varphi(\sigma) = \sigma$ and φ reverses the orientation of σ .*

Proof. We fix an oriented Ptolemy circle $\sigma \subset X$ and distinct points $x, y \in \sigma$. For an oriented circle $\sigma_0 \subset X$ we denote by A the set of all the circles $\varphi(\sigma_0)$, $\varphi \in \text{Aut } X$, with the induced orientation which pass also through x and y . Let

$$\alpha = \inf\{\text{slope}(\sigma, \sigma') : \sigma' \in A\}.$$

By two-point homogeneity property, see Proposition 3.15, $A \neq \emptyset$. Applying Lemma 7.7 we find $\sigma' \in A$ with $\text{slope}(\sigma, \sigma') = \alpha$. Next, we show that $\alpha < 1$. Since $X \neq \widehat{\mathbb{R}}$, there is a shift which makes σ_0 disjoint with σ . Taking a point $\omega \in \sigma_0$ as infinitely remote, we consider all Ptolemy lines in X_ω which are Busemann parallel to $\sigma_0 \setminus \omega$ and intersect σ . Since σ is bounded in X_ω , at least one of them, l , is not tangent to σ . Then $\text{slope}(\sigma, l) < 1$. This l can be obtained from $\sigma_0 \setminus \omega$ by a shift. Applying another shift to l in the space $X_{\omega'}$ with $\omega' \in \sigma \cap l$ (this does not change the slope), we can assume that $x \in \sigma \cap l$.

Repeating this in the space X_x , we find $\tilde{\sigma} \in A$ with $\text{slope}(\sigma, \tilde{\sigma}) < 1$. Thus $\alpha < 1$.

We show that $\alpha = -1$. Then $\sigma = \varphi(\sigma_0)$ as oriented Ptolemy circles for some $\varphi \in \text{Aut } X$, which would complete the proof.

Assume that $\alpha > -1$. The points x, y subdivide each of the circles σ, σ' into two arcs. We choose an arc $\sigma_+ \subset \sigma$ leading from x to y according to the orientation of σ , and an arc $\sigma'_+ \subset \sigma'$ leading from y to x according to the orientation of σ' . Taking a point $\omega \in \sigma'$ inside of the opposite to σ'_+ arc, we see that $l = \sigma' \setminus \omega$ is a Ptolemy line in the space X_ω oriented from y to x .

Given $x' \in \sigma_+$, for every Ptolemy line $l_{x'} \subset X_\omega$ through x' , which is Busemann parallel to l and is oriented as l , we have $\text{slope}(\sigma, l_{x'}) \geq \alpha$ by the definition of α , because by the same argument as above $l_{x'}$ can be put in the set A without changing the slope.

Let $b^\pm : X_\omega \rightarrow \mathbb{R}$ be the opposite Busemann functions of l normalized by $b^+(x) = 0, b^+(y) = -a, b^-(x) = -a, b^-(y) = 0$, where $a = |xy|$. Using Lemma 7.4 we consider for a sufficiently small $\varepsilon > 0$ arclength parameterizations $c_x, c_y : (-\varepsilon, \varepsilon) \rightarrow \sigma$ with $c_x(0) = x, c_y(0) = y, c_x(s), c_y(s) \in \sigma_+$ for $s > 0$, of neighborhoods of x, y respectively in σ . Since Busemann functions on X_ω are affine and hence differentiable along Ptolemy lines, and since the derivative $\frac{db^+ \circ c_x}{ds}(s)$ coincides with the derivative of b^+ along the tangent line to σ at $c_x(s)$, we have

$$\frac{db^+ \circ c_x}{ds}(s) = \text{slope}(\sigma, l_{c_x(s)}) \geq \alpha.$$

For a sufficiently small $t > 0$ let $x_t \in \sigma_+$ be a point at the distance t from x , $|x_t x| = t, x_t = c_x(\tau)$ for some $\tau = \tau(t)$. By integrating we obtain $b^+(x_t) \geq \alpha L(xx_t) \geq \alpha t$. A similar argument shows that $b^-(y_t) \geq \alpha t$, where $y_t = c_y(\tau')$ for some $\tau' = \tau'(t), |yy_t| = t$. Therefore, $b^+(x_t) + b^-(y_t) \geq 2\alpha t$. This contradicts the estimate (9) of Lemma 7.5. Thus $\alpha = -1$. \square

Proof of Proposition 7.1. Given a Möbius map $\psi : \sigma \rightarrow \sigma'$ between Ptolemy circles $\sigma, \sigma' \subset X$, we choose orientations of σ, σ' so that ψ preserves the orientations. By Proposition 7.8 there is $\varphi \in \text{Aut } X$ with $\varphi(\sigma) = \sigma'$ preserving the orientations. Then $\varphi^{-1} \circ \psi : \sigma \rightarrow \sigma$ preserves the orientation of σ , and hence it extends by Lemma 7.6 to $\varphi' \in \text{Aut } X, \varphi'|_\sigma = \varphi^{-1} \circ \psi$. Then $\varphi \circ \varphi' \in \text{Aut } X$ is a required Möbius automorphism. \square

8 Filling of X

We assume that a Möbius space X satisfies the assumptions of Theorem 1.1, and furthermore that $\dim X \geq 2$, since in the case $\dim X = 1$ the space X is Möbius equivalent to $\widehat{\mathbb{R}} = \partial_\infty \mathbb{H}^2$.

8.1 Basic facts

We introduce the notion of a strict space inversion, or an ss-inversion for brevity. A space inversion $\varphi_a : X \rightarrow X$, $a = (\omega, \omega', S)$, is said to be *strict*, if for any $x, x' \in X$ with $\varphi_a(x) = x'$ there exists a sphere T between x, x' such that $\varphi_a = \varphi_b$, where $b = (x, x', T)$.

The following lemma is a modification of Lemma 3.3.

Lemma 8.1. *Let $\varphi : X \rightarrow X$ be a Möbius involution, $\varphi^2 = \text{id}$, of a Ptolemy space X with $\varphi(\omega) = \omega'$ for distinct $\omega, \omega' \in X$. Then there is a unique sphere $S \subset X$ between ω, ω' invariant for φ , $\varphi(S) = S$.*

Proof. Let d be a metric of the Möbius structure with infinitely remote point ω' . Since $\varphi(\omega) = \omega'$, the point ω is infinitely remote for the induced metric φ^*d . Thus for some $\lambda > 0$ we have

$$(\varphi^*d)(x, y) = \frac{\lambda d(x, y)}{d(x, \omega)d(y, \omega)}$$

for each $x, y \in X$ which are not equal to ω simultaneously. We let $S = S_r^d(\omega) \subset X$ be a metric sphere between ω, ω' with $r^2 = \lambda$, $d(x, \omega) = r$ for every $x \in S$. Then

$$d(\varphi(x), \omega) = d(\varphi(x), \varphi(\omega')) = (\varphi^*d)(x, \omega') = \lambda/d(x, \omega) = r$$

for every $x \in S$. Hence $\varphi(S) = S$. For any $x \in X$ with $d(x, \omega) \leq r$ the same argument shows that $d(\varphi(x), \omega) \geq r$, thus an invariant sphere between ω, ω' is unique. \square

Proposition 8.2. *Every s-inversion $\varphi : X \rightarrow X$ is strict.*

Proof. Let $\varphi = \varphi_a$ for $a = (\omega, \omega', S)$ and assume that $\varphi_a(x) = x'$ for some $x, x' \in X$. By Lemma 8.1, there is a unique sphere T between x, x' with $\varphi_a(T) = T$. We put $b = (x, x', T)$ and consider the s-inversion $\varphi_b : X \rightarrow X$. Then the s-inversions φ_a, φ_b both permute x, x' and preserve the sphere T . In addition, φ_b preserves every Ptolemy circle through x, x' . We show that φ_a also preserves every Ptolemy circle through x, x' . This would imply $\varphi_b = \varphi_a$ by uniqueness from the property (I), i.e. that φ is strict.

Let $D = D_{x'} \subset X_x$ be the fiber through x' of the canonical distribution on X_x , that is, D consists of all Ptolemy lines in X_x through x' . Since φ_b preserves every Ptolemy circle through x, x' , the intersection $T \cap D$ is φ_b -invariant, $\varphi_b(T \cap D) = T \cap D$, and moreover $y, x', \varphi_b(y)$ lie on a Ptolemy line through x' in this order for every $y \in T \cap D$. Thus $\varphi_b|_{T \cap D}$ induces the antipodal involution $\overline{\varphi}_b : S_r(o) \rightarrow S_r(o)$ of the sphere $S_r(o) \subset B_x$ of radius r centered at $o = \pi_x(x')$ in the Euclidean space B_x , where r is the radius of the sphere T in the metric of X_x , $T = \{y \in X_x : |yx'| = r\}$. In particular, $\overline{\varphi}_b$ is an isometry of $S_r(o)$.

On the other hand, the composition $\mu = \varphi_a \circ \varphi_b : X \rightarrow X$ preserves x, x' and T , hence $\mu : X_x \rightarrow X_x$ is an isometry. Thus it projects to an isometry $\bar{\mu} : B_x \rightarrow B_x$ of the base. Since $\varphi_a = \mu \circ \varphi_b$ and both μ, φ_b project to isometries of $S_r(o)$, we see that $\varphi_a|_{T \cap D}$ projects to an isometry $\bar{\varphi}_a = \bar{\mu} \circ \bar{\varphi}_b$ of $S_r(o)$. Moreover, $\bar{\varphi}_a$ is an involution, $\bar{\varphi}_a^2 = \text{id}$, because φ_a is. However, every isometric involution of $S_r(o)$, obviously, preserves every pair of antipodal points. This means that φ_a preserves every Ptolemy circle through x, x' . \square

The set Y of all the s-inversions is called the *filling* of X , $Y = \text{Fill } X$. Any pair $\omega, \omega' \in X$ of distinct points determines a *line* in Y : a point $t \in Y$ lies on a line (ω, ω') iff $t(\omega) = \omega'$. In particular, two lines $(\omega, \omega'), (x, x')$ in Y intersect if there is $t \in Y$ such that $t(\omega) = \omega', t(x) = x'$.

The Möbius group of X naturally acts on Y by conjugation: for every Möbius automorphism $g : X \rightarrow X$ we have

$$g^*(t) = g \circ t \circ g^{-1}$$

for the induced $g^* : Y \rightarrow Y$. In particular, $t \in Y$ is fixed for g^* iff $g \circ t = t \circ g$.

The following lemma describes the upper half-space model of $\text{Fill } X$.

Lemma 8.3. *For a fixed $\omega \in X$ and every $t \in \text{Fill } X$ there exist a uniquely determined sphere S between $\omega, t(\omega)$ such that $t = \varphi_b$ for $b = (\omega, t(\omega), S)$. Thus for a fixed metric on X_ω , the space $Y = \text{Fill } X$ is canonically identified with $Y = X_\omega \times \mathbb{R}_+$ by $t = (t(\omega), r)$, where $r > 0$ is the radius of the sphere S . In particular, Y is a smooth manifold diffeomorphic to \mathbb{R}^{n+1} , $n = \dim X$.*

Proof. By Proposition 8.2, there is a uniquely determined sphere S between $\omega, \omega' = t(\omega)$ with $\varphi_b = t$, where $b = (\omega, \omega', S)$. Thus $Y = X_\omega \times \mathbb{R}_+$. By Corollary 6.4, the group N_ω acting on X_ω simply transitively is a Lie group diffeomorphic to \mathbb{R}^n . Therefore, Y is a smooth manifold diffeomorphic to \mathbb{R}^{n+1} . \square

Lemma 8.4. *Assume a space inversion $s \in \text{Fill } X$ has representations $s = \varphi_a = \varphi_b$ for $a = (\omega, \omega', S)$, $b = (\xi, \xi', T)$, where $S, T \subset X$ are spheres between ω, ω' and ξ, ξ' respectively. Then the intersection $S \cap T$ is not empty, $S \cap T \neq \emptyset$.*

Proof. The intersection $S \cap T$ is not empty (and invariant under s) because the spheres S, T are invariant under s , connected (see Corollary 6.4), and s has no fixed point in X , thus it permutes the components of $X \setminus S$ (as well as those of $X \setminus T$). \square

8.2 Lines in the filling

We let $Y = \text{Fill } X$.

Lemma 8.5. *Every line $\gamma = (a, a') \subset Y$ is homeomorphic to \mathbb{R} .*

Proof. By definition, $\gamma = \{t \in Y : t(a) = a'\}$. In the upper half-space model $Y = X_a \times \mathbb{R}_+$, we have $\gamma = \{a'\} \times \mathbb{R}_+ \simeq \mathbb{R}$ because every sphere $S \subset X_\omega$ between a and a' is centered at a' . \square

Proposition 8.6. *Through any two distinct point $s, t \in Y$ there is a unique line $\gamma = (a, a') \subset Y$.*

For a fixed $t \in Y$ we define a map $h : Y \rightarrow \text{Aut } X$ by $s \mapsto h_s = s \circ t : X \rightarrow X$. Note that $h_t = \text{id}$. We put $Y_t = Y \setminus t$ and

$$A = \{s \in Y_t : s \text{ and } t \text{ have a common pair of antipodal points in } X\}.$$

An equivalent definition is

$$A = \{s \in Y_t : h_s \text{ has a fixed point in } X\}.$$

Lemma 8.7. *The set Y_t is connected, and $A \subset Y_t$ is nonempty and closed in Y_t .*

Proof. We fix $\omega \in X$ and a metric of the Möbius structure on X_ω . Then by Lemma 8.3, we have an upper half-space model $Y = X_\omega \times \mathbb{R}_+$. By Corollary 6.4, X_ω is homeomorphic to \mathbb{R}^n with $n \geq 1$. Thus Y_t is connected.

We have $t = (a, R) \in X_\omega \times \mathbb{R}_+$, where R is the radius of the t -invariant sphere $S_t \subset X_\omega$ between a and ω , $S_t = \{x \in X_\omega : |xa| = R\}$. Then for every $s = (a, r) \in Y$, $r \neq R$, the composition $h_s = s \circ t : X_\omega \rightarrow X_\omega$ is a nontrivial pure homothety with $h_s(a) = a$, see Proposition 3.6. Thus $s \in A$, and A is nonempty.

Assume $\lim_i s_i = s \in Y_t$ for a sequence $s_i \in A$. For every i there is $a_i \in X$ with $s_i \circ t(a_i) = a_i$. Since X is compact, the sequence a_i subconverges to $a \in X$. We put $\omega := t(a)$. Then $t = (a, R)$ for some $R > 0$ in the upper half-space model $Y = X_\omega \times \mathbb{R}_+$. We also have $s = (b, r) \in X_\omega \times \mathbb{R}_+$. To prove that $s \in A$ it suffices to show that $b = a$. We have $s_i = (b_i, r_i)$ with $b_i \rightarrow b$, $r_i \rightarrow r$. Note that $s_i(a_i) = t(a_i) \rightarrow \omega$, i.e. the sequence $s_i(a_i)$ tends to infinity in the space X_ω . But if $b \neq a$, the sequence b_i is separated from a , and since $r_i < 2r$, the sequence $s_i(a_i)$ is bounded in X_ω , $|s_i(a_i)b| \leq 4r^2/|ab|$ for all sufficiently large i . This contradiction shows that $s \in A$ and thus A is closed in Y_t . \square

Lemma 8.8. $A = Y_t$.

Proof. In view of Lemma 8.7, it suffices to check that A is open in Y_t . If not, there is sequence $C \ni s_i \rightarrow s \in A$, $C = Y_t \setminus A$. Then s, t have a common pair o, ω of antipodal points in X , $s(o) = t(o) = \omega$, and $h_s = s \circ t : X_\omega \rightarrow X_\omega$ is a pure homothety with coefficient $\lambda \neq 1$ centered at o , $h_s(o) = o$. We can

assume W.L.G. that $0 < \lambda < 1$. By definition, $h_i = h_{s_i} : X \rightarrow X$ has no fixed point in X for every i .

Let $B = B_r(o) \subset X_\omega$ be a metric ball centered at o , $B_r(o) = \{x \in X_\omega : |xo| \leq r\}$ for some $r > 0$. Then $h_s(B) \subset \text{int } B$. Since $h_i \rightarrow h_s$ as $i \rightarrow \infty$ in the Möbius group of X , one easily checks that $h_i(B) \subset B$ for every sufficiently large i . By Corollary 6.4, B is homeomorphic to the standard ball in \mathbb{R}^n . Thus h_i has a fixed point in B for all sufficiently large i . This contradicts the assumption $s_i \in C$. Hence $A = Y_t$. \square

Proof of Proposition 8.6. By Lemma 8.8, the space inversions $s, t : X \rightarrow X$ have a common pair (a, a') of antipodal points, $s(a) = a' = t(a)$. It means that the line $\gamma = (a, a') \subset Y$ passes through s, t .

Assume there is another line $\gamma' = (b, b')$ through s, t . Then a, a', b, b' are fixed points of the composition $h = s \circ t$. In particular, h acts on X_a as a homothety with coefficient $\lambda \neq 1$. But any such homothety of X_a has a unique fixed point. Thus we have W.L.G. $b = a, b' = a'$, that is, $\gamma = \gamma'$. \square

8.3 Distance in the filling

Let $Y = \text{Fill } X$ be the filling of X . Given a line $\gamma = (\omega, \omega') \subset Y$ and points $s, t \in \gamma$, let $S, T \subset X$ be spheres between ω, ω' such that $s = \varphi_a, t = \varphi_b$ for $a = (\omega, \omega', S), b = (\omega, \omega', T)$. We pick $x \in S, y \in T$ and set

$$\rho(s, t) = |\ln \langle \omega, x, y, \omega' \rangle|,$$

where $\langle \omega, x, y, \omega' \rangle = \frac{|\omega y| \cdot |x \omega'|}{|\omega x| \cdot |y \omega'|}$ is the cross-ratio of the quadruple (ω, x, y, ω') . This definition of the distance ρ in Y is independent of the choice $x \in S, y \in T$. Indeed, let $r > 0$ be the radius of the sphere S (centered at ω') w.r.t. a metric in X_ω , and R the radius of the sphere T . Then $\langle \omega, x, y, \omega' \rangle = |x \omega'| / |y \omega'| = r/R$ and $\rho(s, t) = |\ln(r/R)|$. It follows from Proposition 8.6 that $\rho(s, t) > 0$ is well defined for each distinct $s, t \in Y$, and we set $\rho(s, s) = 0$ for every $s \in Y$. Furthermore, the distance ρ is symmetric, $\rho(s, t) = \rho(t, s)$. However, it is not at all obvious that it satisfies the triangle inequality. Nevertheless, the Möbius group of X acts on Y via conjugation, see sect. 8.1, by ρ -isometries because Möbius transformations preserve the cross-ratio.

Lemma 8.9. *Every line $\gamma = (a, a') \subset Y$ is a ρ -geodesic,*

$$\rho(s_1, s_3) = \rho(s_1, s_2) + \rho(s_2, s_3)$$

for any $s_1, s_2, s_3 \in \gamma$ in this order.

Proof. By Lemma 8.5, $\gamma = \{a'\} \times \mathbb{R}_+$ in the upper half-space model $Y = X_a \times \mathbb{R}_+$. Thus $s_i = (a', r_i)$, and we assume W.L.G. that $r_1 \leq r_2 \leq r_3$. Then $\rho(s_i, s_j) = \ln \frac{r_j}{r_i}$ for $i < j$. Hence, the claim. \square

8.4 Embeddings of H^2 into $\text{Fill } X$

Proposition 8.10. *For every Ptolemy circle $\sigma \subset X$ the set*

$$Y_\sigma = \{s \in Y = \text{Fill } X : s(\sigma) = \sigma\}$$

is ρ -isometric to the hyperbolic plane H^2 .

Proof. We fix $\omega \in \sigma$ and show that in the upper half-space model $Y = X_\omega \times \mathbb{R}_+$ the set Y_σ coincides with $\sigma_\omega \times \mathbb{R}_+$, where $\sigma_\omega = \sigma \setminus \{\omega\} \subset X_\omega$ is a Ptolemy line. Indeed, every $s \in Y$ of type $s = (o, r)$ with $o \in \sigma_\omega$ preserves σ because s permutes $o, \omega \in \sigma$, i.e. $\sigma_\omega \times \mathbb{R}_+ \subset Y_\sigma$. In the opposite direction, we have $o = s(\omega) \in \sigma_\omega$, thus $s = (o, r) \in \sigma_\omega \times \mathbb{R}_+$ for some $r > 0$.

Since $s, t \in Y_\sigma$ preserve σ , we have $o, \omega \in \sigma$ for the line $(o, \omega) \subset Y$ through s, t . Then the ρ -distance between s, t can be computed in the upper half-space model $X_\omega \times \mathbb{R}_+$ as $\rho(s, t) = |\ln \frac{R}{r}|$, where $s = (o, r), t = (o, R)$. But we have exactly this formula in the upper half-space model for H^2 . Therefore $Y_\sigma = \sigma_\omega \times \mathbb{R}_+$ is ρ -isometric to H^2 . \square

8.5 Semi-norm on the tangent bundle to the filling

Recall that $Y = \text{Fill } X$ is a smooth manifold, see Lemma 8.3, and thus the tangent space $T_s Y$ to Y is well defined for every $s \in Y$. For every $a \in X$ the line $(a, a') \subset Y$, $a' = s(a)$, passes through s by definition. Then the speed vector $v = dh(\frac{d}{d\tau})$ at $\tau = 0$ of some smooth parameterization $h : \mathbb{R} \rightarrow Y$, $h(0) = s$, of the line (a, a') is tangent to Y at s , $v \in T_s Y$. This defines a map $f : X \rightarrow S_s Y$ to the space $S_s Y$ of oriented directions in $T_s Y$, which is continuous (to see the continuity one should think of the line $(a, a') \subset Y$ through s as the set of homotheties $\Gamma_{a, a'} \subset \text{Aut } X$ by composing every $s' \in (a, a')$ with s). By Proposition 8.6, the set of directions $A \subset S_s Y$ tangent to the lines $(a, s(a))$, $a \in X$, is dense in $S_s Y$. Then $A = S_s Y$ because X is compact and hence the set $A = f(X)$ is closed in $S_s Y$.

Thus any nonzero vector $v \in T_s Y$ is tangent to a line (a, a') in Y through s , where $a = a(v)$, $a' = s(a) \in X$. We assume that $s = h(0)$ for some smooth parameterization $h : \mathbb{R} \rightarrow Y$ of the line (a, a') with $v = dh(\frac{d}{d\tau}) = dh/d\tau$. Then the formula

$$\|v\| := \lim_{\tau \rightarrow 0} \frac{1}{|\tau|} \rho(h(\tau), s)$$

defines a semi-norm on $T_s Y$, because it follows from the definition that $ds(v) = -v$ and $\| -v \| = \| ds(v) \| = \| v \|$. The semi-norm $\|v\|$ is called the ρ -semi-norm.

For example, in the upper half-space model $Y = X_{a'} \times \mathbb{R}_+$, we have $s = (a, u)$ for some $u > 0$. Consider a parameterization $h = h_\alpha : \mathbb{R} \rightarrow Y$ of the line (a, a') with $h(0) = s$ given by $h(\tau) = (a, ue^{\alpha\tau})$ for some $\alpha > 0$. Then

$$\|dh/d\tau\| = \lim_{\delta \rightarrow 0} \frac{1}{|\delta|} \rho(h(\tau + \delta), h(\tau)),$$

and we have $\rho(h(\tau), h(\tau + \delta)) = |\ln(R/r)|$ for $R = ue^{\alpha(\tau+\delta)}$, $r = ue^{\alpha\tau}$. Thus $\|dh/d\tau\| = \alpha$ for every $\tau \in \mathbb{R}$, and h is a constant ρ -speed parameterization. There is $\alpha > 0$ such that v is a speed vector of $h = h_\alpha$, hence $\|v\| = \alpha$.

We identify the direction space $S_s Y$ with the ρ -unit sphere, $S_s Y = \{v \in T_s Y : \|v\| = 1\}$.

Lemma 8.11. *For every $s \in Y$ the stabilizer of s in the ρ -isometry group of Y acts transitively on the ρ -unit sphere $S_s Y \subset T_s Y$.*

Proof. Given $v, v' \in S_s Y$ we let $(a, a'), (b, b') \subset Y$, $a, b \in X$, $a' = s(a)$, $b' = s(b)$, be the lines through s with tangent vectors v, v' at s respectively. There are Ptolemy circles $\sigma, \sigma' \subset X$ through a, a' and b, b' respectively. The subspaces $Y_\sigma, Y_{\sigma'} \subset Y$ isometric to \mathbb{H}^2 contain s and their tangent spaces $H_\sigma, H_{\sigma'} \subset T_s Y$ contain v, v' respectively. There is a (uniquely determined up to the reflection in the line $(a, a') \subset Y_\sigma$) Möbius automorphism $\varphi : \sigma \rightarrow \sigma'$ with $\varphi(a) = b$, $\varphi(a') = b'$ such that its extension to Y_σ is an isometry $\varphi : Y_\sigma \rightarrow Y_{\sigma'}$ with $\varphi(s) = s$. Then $d\varphi(v) = v'$.

By Proposition 7.1, the space X possesses the extension property (E₂), which implies that $\varphi : \sigma \rightarrow \sigma'$ extends to a Möbius $\varphi : X \rightarrow X$ and therefore to a ρ -isometry $\varphi : Y \rightarrow Y$ with $\varphi(s) = s$. Thus the stabilizer of s in the ρ -isometry group of Y acts transitively on $S_s Y$. \square

Proposition 8.12. *For every $s \in Y$, the ρ -semi-norm on the tangent space $T_s Y$ is an Euclidean norm.*

Proof. Let $B \subset T_s Y$ be the ρ -unit ball centered at 0. We first show that B is convex and, hence, the ρ -semi-norm is a norm. Since $\|v\| > 0$ for every nonzero $v \in T_s Y$, B is compact and thus there is a supporting affine hyperplane $H \subset T_s Y$, i.e. H touches B such that B is contained in one of half-spaces determined by H . By Lemma 8.11, every boundary points of B possesses this property and thus B coincides with the intersection of half-spaces, i.e. B is convex.

With the transitivity of the isometry action, it is well known that then B is an ellipsoid and hence the ρ -norm is an Euclidean one. (We briefly sketch the argument. It is well known that every isometry of $T_s Y$ is affine. Hence, the group K of linear automorphisms of $T_s Y$ that fix 0 and preserves B acts transitively on ∂B . Fixing a background Euclidean metric on $T_s Y$, we find the Löwner ellipsoid L inscribed in B , that is, the ellipsoid of maximal volume. This ellipsoid is unique and invariant under K . Thus $B = L$). \square

Corollary 8.13. *The filling $Y = \text{Fill } X$ is a rank one symmetric space of non-compact type w.r.t. the metric ρ such that $\partial_\infty Y = X$.*

Proof. The Riemannian metric on Y obtained in Proposition 8.12 is smooth because the isometry group of Y , which acts transitively on Y , is a Lie group. Therefore Y is a Riemannian symmetric space of non-compact type.

We check that there is no Euclidean geodesic plane $E \subset Y$, $\dim E = 2$. Assume that Y contains such an E . By Proposition 8.6, for each distinct $s, t \in E$ there exists a line $\gamma = (a, a')$ through s, t . Then $\gamma \subset E$ is invariant for $h = s \circ t$ and $h(a) = a, h(a') = a'$. For any $u \in E$ the line $\gamma' = u(\gamma) \subset E$ is parallel to γ because u acts on E as a central symmetry. However every line $\gamma' \subset E$ parallel to γ is also invariant for h , $h(\gamma') = \gamma'$. Thus $\gamma' = (a, a')$ because a, a' are only points in X fixed by h . By Lemma 8.5, the line $(a, a') \subset Y$ is homeomorphic to \mathbb{R} and uniquely determined by the end points a, a' . Hence $\gamma' = \gamma$, and u preserves γ , $u(\gamma) = \gamma$. Then $u \in \gamma$ by definition. This is a contradiction because $\dim E = 2$ and $E \neq \gamma$. It follows that the sectional curvatures of Y are negative. Then the Gromov boundary $\partial_\infty Y$ coincides with the geodesic boundary of Y which is X by definition of geodesic lines in Y . Thus $\partial_\infty Y = X$. \square

8.6 Horospheres in the filling

In this section we show that the Riemannian distance ρ on the filling $Y = \text{Fill } X$ induces the initial Möbius structure of X .

By Proposition 8.18, the Riemannian metric associated with the distance ρ has negative sectional curvatures. Thus every parabolic isometry of Y preserving $\omega \in X$ leaves invariant every horosphere in Y centered at ω . For example, the group N_ω of X_ω -shifts acts on Y by parabolic isometries preserving ω .

We fix $\omega \in X$ and a metric $|xy| = |xy|_\omega$ on X_ω . Let $Y = X_\omega \times \mathbb{R}_+$ be the respective upper half-space model of the filling Y . Every space inversion $t \in Y$ is represented as $t = (t(\omega), r_t) \in X_\omega \times \mathbb{R}_+$, where $r_t > 0$ is the radius of the t -invariant sphere between $t(\omega)$ and ω . Since the group N_ω acts on X_ω by isometries, it leaves invariant every set $H_r = X_\omega \times \{r\} \subset Y$.

Let $b : Y \rightarrow \mathbb{R}$ be the Busemann function associated with a geodesic ray $\gamma_a = \{a\} \times [R, \infty) \subset X_\omega \times \mathbb{R}_+$, $b(s) = 0$ for $s = (a, R) \in Y$. Then $b(t) = \pm \rho(s, t) = \ln \frac{R}{r_t}$ for every $t = (a, r_t) \in Y$. By the remark above, the horosphere $b^{-1}(r)$ of b is N_ω -invariant, thus $b^{-1}(r) = H_r$ for every $r \in \mathbb{R}$ because N_ω is transitive on X_ω . It follows that any function $b : Y \rightarrow \mathbb{R}$ of type

$$b(t) = \ln \frac{R}{r_t}$$

for $t = (t(\omega), r_t)$, $R > 0$, is a Busemann function in the upper half-space model $Y = X_\omega \times \mathbb{R}_+$ centered at ω .

Lemma 8.14. *Assume that distinct lines $(a, a'), (\omega, \omega') \subset Y$ intersect at $s \in Y$, $s = (a, a') \cap (\omega, \omega') \in H_r$ for some $r > 0$. Then*

$$|x\omega'| \cdot |s(x)\omega'| = r^2$$

for every $x \in X \setminus \{\omega, \omega'\}$. In particular, $|a\omega'| \cdot |a'\omega'| = r^2$.

Proof. By Proposition 8.2, the space inversion s can be represented as $s = \varphi_c$ for $c = (\omega', \omega, T)$, where T is the sphere between ω' , ω of radius r in X_ω . Since $\omega' = s(\omega)$, we obtain, using Lemma 3.3 applied to s , that $|s(x)\omega'| = |s(x)s(\omega)| = \frac{r^2}{|x\omega'|}$ for every $x \in X \setminus \{\omega, \omega'\}$, \square

Lemma 8.15. *Given line $\gamma = (a_0, a_1) \subset Y$ and its points $s_0 = (\omega_0, r_0)$, $s_1 = (\omega_1, r_1) \in \gamma$ with distinct $\omega_0, \omega_1 \in X_\omega$ in the order a_0, s_0, s_1, a_1 , we have*

$$|a_i\omega_i| < \frac{4r_i^2}{|\omega_0\omega_1|} \quad (10)$$

for $i = 0, 1$ and $\max\{r_0, r_1\} \leq |\omega_0\omega_1|/4$.

Proof. The space inversion s_i lies on the lines (a_0, a_1) and (ω, ω_i) in Y , $i = 0, 1$. Thus by Lemma 8.14,

$$|a_0\omega_i| \cdot |a_1\omega_i| = r_i^2. \quad (11)$$

We have $|\omega_0\omega_1| \geq 4r$ for $r = \max\{r_0, r_1\}$, thus $\text{dist}(B_{r_0}(\omega_0), B_{r_1}(\omega_1)) \geq |\omega_0\omega_1| - (r_0 + r_1) \geq 2r$ for the balls in X_ω of radii r_0, r_1 centered at ω_0, ω_1 respectively. Therefore, one of the points a_0, a_1 lies in $B_{r_0}(\omega_0)$, while the other in $B_{r_1}(\omega_1)$. We show $|a_i\omega_i| < r_i$ for $i = 0, 1$.

Indeed, let $S_i, T_i \subset X$ be the spheres between a_0, a_1 , respectively ω, ω_i through s_i , $i = 0, 1$. The order a_0, s_0, s_1, a_1 on l tells us that the cross-ratio

$$\langle a_0, x_0, x_1, a_1 \rangle > 1$$

for any $x_0 \in S_0, x_1 \in S_1$.

By Lemma 8.4 there is $x_i \in S_i \cap T_i$. The assumption $|a_0\omega_1| < r_1$ implies $|a_1\omega_0| < r_0$. Then $|a_0x_0|, |a_1x_1| > \text{dist}(B_r(\omega_0), B_r(\omega_1)) \geq 2r$ and $|a_0x_1| < 2r_1, |a_1x_0| < 2r_0$. Therefore

$$\langle a_0, x_0, x_1, a_1 \rangle = \frac{|a_0x_1| \cdot |a_1x_0|}{|a_0x_0| \cdot |a_1x_1|} < \frac{4r_0r_1}{4r^2} \leq 1,$$

a contradiction. That is, $a_i \in B_{r_i}(\omega_i)$ for $i = 1, 2$.

We have $|a_0a_1| \geq |\omega_0\omega_1| - 2r \geq 2r$. By the triangle inequality, $|a_i\omega_j| \geq |a_0a_1| - |a_j\omega_j|$. Using (11), we obtain $|a_i\omega_i|(|a_i\omega_i| - |a_0a_1|) + r_i^2 \geq 0$ for $i = 0, 1$. Solving this quadratic inequality, we find

$$|a_i\omega_i| \leq \frac{|a_0a_1|}{2} \left(1 - \sqrt{1 - \frac{4r_i^2}{|a_0a_1|^2}} \right) < \frac{2r_i^2}{|a_0a_1|} \leq \frac{4r_i^2}{|\omega_0\omega_1|},$$

because $|\omega_0\omega_1| \leq |a_0a_1| + 2r \leq 2|a_0a_1|$. \square

Proposition 8.16. *Given $s_0 = (\omega_0, r_0)$, $s_1 = (\omega_1, r_1) \in Y$ with distinct $\omega_0, \omega_1 \in X_\omega$, we have*

$$\exp\left(\frac{1}{2}\rho(s_0, s_1)\right) = \frac{|\omega_0\omega_1|}{\sqrt{r_0r_1}}(1 + O(r)),$$

where for $r = \max\{r_0, r_1\} \leq |\omega_0\omega_1|/4$ the function $O(r)$ can be estimated as $|O(r)| \leq \frac{c_0r}{|\omega_0\omega_1|}$ with some universal constant c_0 .

Proof. Let $\gamma = (a_0, a_1)$ be the line through s_0, s_1 with the order a_0, s_0, s_1, a_1 of our points. By Proposition 8.2, the space inversion s_i can be represented as $s_i = \varphi_{b_i} = \varphi_{c_i}$ for $b_i = (a_0, a_1, S_i)$, $c_i = (\omega, \omega_i, T_i)$, where S_i is a sphere between a_0, a_1 , and T_i a sphere between ω, ω_i , $i = 0, 1$. By the assumption, T_i is the sphere in X_ω of radius r_i centered at ω_i . By Lemma 8.4 there is $x_i \in S_i \cap T_i$. Then $|x_i a_i| = |x_i \omega_i| + O(r_i^2) = r_i + O(r_i^2)$ with $|O(r_i^2)| < \frac{4r_i^2}{|\omega_0\omega_1|}$ according (10). Next, for $i \neq j$ we have $|x_i a_j| = |\omega_0\omega_1| + O_i(r_0, r_1)$ with $|O_i(r_0, r_1)| \leq r_i + |a_j \omega_j| < r + \frac{4r^2}{|\omega_0\omega_1|}$ again by (10). We conclude that

$$\frac{|x_i a_j|}{|x_i a_i|} = \frac{|\omega_0\omega_1|}{r_i} (1 + O(r)),$$

where $|O(r)| \leq \frac{c_1r}{|\omega_0\omega_1|}$ for some universal constant c_1 .

We have $\rho(s_0, s_1) = |\ln \langle a_0, x_0, x_1, a_1 \rangle|$ for any $x_i \in S_i$, and

$$\langle a_0, x_0, x_1, a_1 \rangle = \frac{|a_0 x_1| \cdot |x_0 a_1|}{|a_0 x_0| \cdot |x_1 a_1|} = \frac{|\omega_0\omega_1|^2}{r_0 r_1} (1 + O(r)).$$

Thus

$$\exp\left(\frac{1}{2}\rho(s_0, s_1)\right) = \frac{|\omega_0\omega_1|}{\sqrt{r_0 r_1}} (1 + O(r))$$

with $|O(r)| \leq \frac{c_0r}{|\omega_0\omega_1|}$ for some universal constant c_0 . \square

For $s, s' \in Y$ we define their *Gromov product* w.r.t. a Busemann function b by

$$(s|s')_b = \frac{1}{2} (b(s) + b(s') - \rho(s, s')).$$

Corollary 8.17. *For the Busemann function $b : Y \rightarrow \mathbb{R}$ centered at $\omega \in X$ and defined by $b(t) = \ln \frac{1}{r_t}$ for $t = (t(\omega), r_t) \in X_\omega \times \mathbb{R}_+$ and for each distinct $a_0, a_1 \in X_\omega$ there exists a limit*

$$(a_0|a_1)_b := \lim_{r \rightarrow 0} (s_0|s_1)_b \text{ and } e^{-(a_0|a_1)_b} = |a_0 a_1|_\omega,$$

where $s_0, s_1 \in H_r \cap (a_0, a_1)$. In particular, the Riemannian distance ρ on the filling $Y = \text{Fill } X$ induces on $X = \partial_\infty Y$ the initial Möbius structure of X .

Proof. In the upper half-space model for Y with base X_ω , we have $s_i = (\omega_i, r)$ for some $\omega_i \in X_\omega$. We can assume that s_i lie on the line (a_0, a_1) in the order a_0, s_0, s_1, a_1 . By Lemma 8.15, $\omega_i \rightarrow a_i$ as $r \rightarrow 0$. With Proposition 8.16 we have

$$(s_0|s_1)_b = \ln \frac{1}{r} - \frac{1}{2}\rho(s_0, s_1) = -\ln |a_0 a_1|_\omega + o(1)$$

as $r \rightarrow 0$, which implies the corollary. \square

8.7 Sectional curvature of Y

To complete the proof of Theorem 1.1, it remains to check that the Möbius structure of $X = \partial_\infty Y$ is canonical. By Corollary 8.17 it suffices to check that the maximum of sectional curvatures of Y is -1 .

We already mentioned that the assumption (E) of Theorem 1.1 also serves as a normalization condition.

Proposition 8.18. *Sectional curvatures K_σ of Y bounded above by -1 , moreover $\max K_\sigma = -1$.*

Proof. It suffices to check that $K_\sigma \leq -1$ because Y contains geodesic isometric copies of the hyperbolic plane \mathbb{H}^2 . We assume that for every $\omega \in X$ the nilpotent group $N = N_\omega$ of shifts of X_ω is not abelian since otherwise $X = \widehat{\mathbb{R}}^n$ by Proposition 6.6 and $Y = \mathbb{H}^{n+1}$.

We fix $\omega \in X$, $o \in X_\omega$. Then the homothety group $\Gamma = \Gamma_{o,\omega}$ acts on N by conjugation, $\gamma(\eta) = \gamma \circ \eta \circ \gamma^{-1}$ for every $\gamma \in \Gamma$, $\eta \in N$. Thus $G = N \rtimes \Gamma < \text{Aut } X$ is a solvable group of Möbius automorphisms of X , which acts simply transitively by isometries on Y . We fix $s_0 \in Y$, e.g. $s_0 = (o, 1) \in X_\omega \times \mathbb{R}_+$, and identify G with Y by $g \mapsto g^*(s_0) = g \circ s_0 \circ g^{-1}$ for $g \in G$. Then the tangent space $T_{s_0}Y$ is endowed with the Lie algebra structure \mathfrak{g} of G .

Let $H_1 = X_\omega \times \{1\} \subset Y$ be the horosphere through s_0 in the upper half-space model. Then the tangent space $T_{s_0}H_1 \subset T_{s_0}Y$ is identified with the Lie algebra of N , $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{n}$. Let $\{\mathfrak{g}^i\}$ be the lower central series of \mathfrak{n} defined by $\mathfrak{g}^1 = \mathfrak{n}$, $\mathfrak{g}^{i+1} = [\mathfrak{n}, \mathfrak{g}^i]$. Decompose \mathfrak{g}^i orthogonally as $\mathfrak{g}^i = \mathfrak{a}_i + \mathfrak{g}^{i+1}$. Since \mathfrak{n} is nilpotent, we have $\mathfrak{n} = \sum_1^m \mathfrak{a}_i$ for some $m \in \mathbb{N}$. Note that $m \geq 2$ by the assumption that N is not abelian. We fix a unit vector $A_0 \in \mathfrak{g}$ which is orthogonal to \mathfrak{n} . Since Y is symmetric, its curvature tensor \mathcal{R} is parallel, $\nabla \mathcal{R} = 0$. Then by [He] the symmetric part D_0 of the endomorphism $\text{ad } A_0|_{\mathfrak{n}}$ has the eigenvalues $i \cdot \lambda$, $D_0|_{\mathfrak{a}_i} = i \cdot \lambda \cdot \text{id}$ for some $\lambda > 0$ and $i = 1, \dots, m$, see [He, Proposition 3, (iib)]. By [He, Corollary] this implies that nonzero eigenvalues of the curvature operator $u \mapsto \mathcal{R}(u, A_0, A_0)$, i.e. the respective sectional curvatures K_σ , are $-i^2\lambda^2$, $i = 1, \dots, m$. Note that the subspace $\mathfrak{g}^2 = [\mathfrak{n}, \mathfrak{n}] \in T_oX_\omega$ is tangent to the \mathbb{K} -line through o , any nonzero vector $u \in \mathfrak{a}_1 \subset T_oX_\omega$ is tangent to a Ptolemy line in X_ω through o . It means that

the 2-direction $(u, A_0) \subset T_{s_0}Y$ is tangent to a geodesic isometric copy of H^2 in Y . Thus $\lambda = 1$ and hence $\max K_\sigma = -1$. \square

Remark 8.19. It is proved in [He, Proposition 3] that actually $m = 2$ and thus Y is 1/4-pinched, $-4 \leq K_\sigma \leq -1$. Again, the proof does not use the classification of rank one symmetric spaces.

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